Advanced Sliding-Mode Control: Theory & Design Techniques

by

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A Thesis submitted in Partial Fulfilment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

1997

In Memory of my absolutely beloved ***Parents***

Nguyễn Tường Diễn

Nguyễn Thị Ngát

who lived and even . . . died for their poor children

Nguyễn Thị Liễu

Nguyễn Thị Nhiễu

Nguyễn Duy Hy

Nguyễn Duy Kỷ

Nguyễn Thị Kim Cương

Nguyễn Thị Tường Vân

NDK

In Memory of all my ***Teachers****,*

all my ***Sisters*** *and* ***Brothers****,*

who had Hard Time for my Good Time.

NDK

Acknowledgments

From the bottom of my heart, I am greatly indebted to my Supervisors, Professor Hung T. Nguyen and Professor Barry S. Thornton, for their sincere discussions and warm supports. I would like to thank Staffs in Control Laboratory, Instrumentation & Control Post-Graduate Research Laboratory at School of Electrical Engineering, University of Technology, Sydney.

I am also greatly grateful to authors mentioned in this thesis whose works have opened my mind for research and development of this work.

Duy-Ky Nguyen

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List of Symbols

$\Re_{(-)} = \{x \in \Re \mid x < 0\}$

$\Re_{(-)}^{1\times n} = \{\mathbf{x} \in \Re^{1\times n} \mid x_i < 0\}$

$\mathbf{x} \in \Re^{n\times 1}$	: system state variable
$\mathbf{A}, \Delta\mathbf{A}, \tilde{\mathbf{A}}, \Delta\tilde{\mathbf{A}} \in \Re^{n\times n}$	: first system matrix
$\mathbf{B}, \Delta\mathbf{B}, \tilde{\mathbf{B}}, \Delta\tilde{\mathbf{B}} \in \Re^{n\times m}$	: second system matrix
$\mathbf{u} \in \Re^{m\times 1}; \quad u \in \Re^{1}$	: control function
$y \in \Re^{1}; \quad \mathbf{y} \in \Re^{p\times 1}$	: output state
$\mathbf{H} \in \Re^{1\times n}$ or $\mathbf{H} \in \Re^{m\times n}$	: hyperplane $s = \mathbf{Hx}$ or $\mathbf{s} = \mathbf{Hx}$

where

$$\mathbf{x} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}\mathbf{u} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).\mathbf{u}$$

$$\mathbf{A} - \Delta\mathbf{A} \le \tilde{\mathbf{A}} \le \mathbf{A} + \Delta\mathbf{A}, \qquad \mathbf{B} - \Delta\mathbf{B} \le \tilde{\mathbf{B}} \le \mathbf{B} + \Delta\mathbf{B}$$

$$\Delta\tilde{\mathbf{A}} \le |\Delta\mathbf{A}|, \qquad \Delta\tilde{\mathbf{B}} \le |\Delta\mathbf{B}|$$

$m_{ij} = \mathbf{M}_{(i,j)}$	: an element at row *i*-th and column *j*-th of a matrix **M**
$\mathbf{M}_{i*}$	**:** row *i*-th of a matrix **M**
$\mathbf{M}_{*j}$	**:** column *j*-th of a matrix **M**
$\mathbf{M}_{(:,j:k)}$	**:** a matrix extracted from column *j*-th to column *k*-th of a matrix
$\lvert\mathbf{M}\rvert = \left[\lvert m_{ij}\rvert\right] \in \Re^{n\times m}$	: absolute of a matrix
$\lvert\mathbf{M}\rvert = \det(\mathbf{M}) \in \Re, \quad \mathbf{M} \in \Re^{n\times n}$	: determinant of a square matrix
$\mathbf{P} = \mathbf{Q} \otimes \mathbf{R}$	: elementwise matrix multiplication, *ie.* $p_{ij} = q_{ij} \times r_{ij}$.

$\boldsymbol{\lambda} = \operatorname{eig}(\mathbf{M}) = \left\{[\lambda_1, \;\cdots,\; \lambda_n] \,\middle|\, \lvert\lambda\mathbf{I} - \mathbf{M}\rvert = 0, \mathbf{M} \in \Re^{n\times n}\right\}$

$\boldsymbol{\lambda}_H$: hyperplane-eigenvalues

$\boldsymbol{\lambda}_S$: sliding-eigenvalues

$\boldsymbol{\lambda}_C$: closed-loop system eigenvalues

$\lvert\mathbf{x}\rvert = \left[\lvert x_i\rvert\right]$	: absolute of a vector
$\lVert\mathbf{x}\rVert = \sqrt{\sum_i x_i^2}$	: Euclidean norm of a vector

$$\operatorname{sgn}(\mathbf{v}) = \operatorname{sgn}[v_1, \;\cdots,\; v_n] = [\operatorname{sgn}(v_1), \;\cdots,\; \operatorname{sgn}(v_n)];$$

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & \text{if } \xi > 0 \\ 0, & \text{if } \xi = 0 \\ -1, & \text{if } \xi < 0 \end{cases}$$

Abstract

In this thesis, advanced design techniques in the sliding-mode control (SMC) are presented. All the proposed designs are developed in a unified manner for robust analog/digital sliding-mode controllers/observers, robust integral SMC, continuous and discontinuous SMC designs, sliding-mode fuzzy control and for MIMO uncertain nonlinear systems. The stability of fuzzy control is developed and sliding-mode fuzzy control is presented on basis of the digital SMC design.

A hyperplane design is fully presented for both eigenvalue allocation and optimal sliding-mode approaches for linear and nonlinear systems. Two eigenvalue types are identified: sliding-eigenvalues and hyperplane-eigenvalues which are as desired sliding-eigenvalues. The concept of sliding-eigenvalues is employed to establish a stability criterion and a sufficient invariance condition.

The proposed analysis of the chattering problem is more comprehensive and precise than the current works. Moreover our approach is in a unified manner for the saturate function and unitvector function. In addition, the introduction of the hyperbolic tangent to replace the saturate and unitvector functions for more convenient and efficient.

In the proposed continuous SMC design, the control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem found in the discontinuous SMC which is known as the variable-structure system (VSS) control. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework. This leads us to propose some novel SMC designs such as a new discontinuous SMC which alleviates the chattering problem, discrete-time SMC with both continuous and discontinuous control functions, continuous-time and discrete-time sliding-mode observers, fuzzy SMC, stability of fuzzy control, etc.

A new robust discrete-time sliding-mode controller-observer design is fully presented for both discontinuous SMC (VSS) and continuous SMC control functions. A high sampling rate is proved to be a necessary condition for a discrete-time SMC; under no uncertainty, this condition reduces to the stability criterion of unit circle in the Z-domain.

For the fuzzy control, we prove that a typical fuzzy rulebase can satisfy the Lyapunov sliding condition so the stability of a fuzzy control is guaranteed by the Lyapunov stability theorem. This can be seen as a stability criterion for the fuzzy control theory.

To design a stable sliding-mode fuzzy controller, a fuzzy mechanism is used to minimize a sliding variable s instead of using the sliding condition as in the sliding mode control. In a typical fuzzy rulebase, it may not be convenient to use more than 2 entries, we can use 1 entry for s and the other for sum of s, and hence a possible steady-state error may be eliminated by this I-action. In a fuzzy control, the problems are how to choose the gains for error and error change and a possible chattering. Using the sliding-mode control theory, these gains can be determined by a hyperplane and the chattering problem can be solved since the system dynamics are taken into account

We develop a new fuzzy identification scheme which is simpler to implement than the current approach. The fuzzy inference is employed to obtain the most potential model from some rough mathematical models from experiments using a proposed practical system identification. A fuzzy model by the proposed scheme can be a solution to the conservative problem.

All theoretical results are consistent with experimental results on Ball-Hoop system, such as infinite gain of the switching function may cause excitation of unmodelled high-frequency; slow-down system responses due to low gain of sliding function; performances of continuous SMC, continuous pseudo-SMC (TanH SMC) and sliding-mode fuzzy control.

Advanced Design Techniques in Sliding-Mode Control: Introduction

1. MOTIVATION

In real dynamical systems, it is impossible to avoid uncertainties due to imperfect modelling, due to the environment such as temperature, pressure... and external disturbances. So the crucial demand is a solution to the *robust control* problem for uncertain systems. One solution to this problem is *H-Infinity Control*, however it can deal with uncertain linear systems only. A more general solution is *Sliding-Mode Control* (SMC) since it is simpler than H-Infinity Control and it can deal with both uncertain linear and nonlinear systems.

2. METHODOLOGY

Our approaches are kept as simple and practically precise as possible compared to complicated and precise approaches as in Ryan *et al* 1987, Spurgeon *et al.* 1993, Edwards *et al.* 1996. Proposed designs are put in theorems for clarity, results can be quickly found in theorems and can be checked in their proofs. All experimental results are consistent with the theoretical results.

3. RESULTS

New robust analog-discrete discontinuous-continuous sliding mode controller-observer designs are presented. The stability of fuzzy control is developed and sliding-mode fuzzy control is presented on basis of the discrete-time SMC design. All the proposed designs are developed in a unified manner for robust analog-discrete-time sliding-mode controllers, robust analog-discrete sliding-mode observers, continuous and discontinuous designs, sliding-mode fuzzy control and for MIMO uncertain nonlinear systems.

In the discontinuous design, the chattering problem is analyzed comprehensively as opposed to the analysis in the current literature. Alternatively to the current literature, our proposed approach is in a unified manner for all sliding functions (saturate, unitvector and hyperbolic tangent) where the hyperbolic tangent function is proposed due to its availability as a standard mathematical function. Based on the sliding-mode mechanism, a design rule is proposed to guarantee an existence of the sliding mode since the invariance property is with this mode. A new reaching control design is proposed to include system dynamics for efficient SMC designs, this is not the case in the current literature.

In the continuous design, the control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem found in the discontinuous sliding-mode control (SMC) which is known as the variable-structure system (VSS) control. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework.

3.1. Chapter 1: Sliding Mode Control Background

In the sliding-mode control theory, control dynamics have 2 sequential modes, the first is the *reaching mode* and the second is the *sliding mode*. The Lyapunov sliding condition forces system states to *reach* a hyperplane and keeps them *sliding* on this hyperplane, so a SMC design is composed of 2 phases, hyperplane design and controller design. First, a hyperplane is designed via the pole-placement approach as in the state-space control, then a controller design is based on the sliding condition. The stability is guaranteed by the sliding condition (Lyapunov Stability Criterion Theorem) *and* by a *stable* hyperplane (stable designer-chosen pole-placement). In the reaching mode, the control dynamics depend on system parameters; but in the sliding mode they depend on the hyperplane, this is the *invariance* property of the sliding mode.

In this chapter, we revise some SMC terminologies such as: hyperplane, sliding condition, sliding margin, equivalent control, the reaching and sliding modes. Some numerical examples are used to illustrate how SMC can handle uncertain systems and non-linear systems.

3.2. Chapter 2: Sliding Hyperplane Design

In this chapter, a hyperplane design is fully presented for both eigenvalue allocation and optimal sliding-mode approaches in a unified manner for linear and nonlinear systems. We identify 2 types of eigenvalues: sliding-eigenvalue and hyperplane-eigenvalue. Sliding-eigenvalues determines the dynamics of the system states in the sliding mode. Hyperplane-eigenvalues are the *desired* sliding-eigenvalues that represent the expected dynamics of the system states in the sliding mode. In addition, the concept of sliding-eigenvalue is conveniently applied in the stability problem and the invariance property (Chapter 3, 4). On the basis of the partition transformation method for *linear systems* in Utkin *et al.* 1978, we propose a direct allocation method that may be the simplest way to see what the invariance property really means (Chapter 3) and extendable into *nonlinear systems* (Chapter 7). Alternatively, we propose a direct calculation method to reduce the computational effort compared to the partition transformation method. A hyperplane normalization is proposed to greatly simplify computation of a control function.

Another approach to design a hyperplane is based on the optimal sliding dynamics: error and error/energy optimizations. To complement the work in the Utkin paper on the error/energy optimization (Utkin *et al.* 1978), proofs of the theorems and a solution to the optimal problem are presented in this chapter.

3.3. Chapter 3: A New Variable-Structure System Controller Design

In this chapter, a Variable-Structure System (VSS) controller design will be fully developed in a unified manner that is extendable into robust control (Chapter 4) and MIMO nonlinear system (Chapter 7). A VSS control is a discontinuous sliding mode control (SMC). Based on the concept of sliding-eigenvalues introduced in Chapter 2, we propose a stability criterion which is much simpler than the current one. By the sliding condition, a control function is first computed in an *if*-form as in Utkin 1977, we then propose a technique to convert this form into a compact form to facilitate controller implementations. We propose a sufficient condition as a design rule to ensure the reaching mode will terminate in a finite time, hence the sliding mode can exist, because the invariance property is with the sliding mode only. A new reaching control design is proposed to include system dynamics for an efficient SMC design, this is not the case in the current literature.

Alternative to the chattering problem analysis in Slotine *et al* 1983, we propose a comprehensive analysis that allows a unified approach for both the saturate (Slotine *et al.* 1983) and unitvector functions (Ryan *et al.* 1984, Spurgeon 1992). In addition, the introduction of the hyperbolic tangent (TanH) to replace the saturate and unitvector functions for more convenient since TanH is a standard mathematical function. The performance of using the sliding functions (saturate, unitvector and hyperbolic tangent functions) in place of the switching function is analyzed in terms of steady-state error, response speed. A design scheme is proposed to cope with limitations arised from these replacements (Example 7.6).

A VSS control is based on a state-space model, an I-action may be required to eliminate a steady-state error. In Chern *et al.* 1991, an integral of error has been used, however it is a pseudo-SMC because the larger controller gain the closer the sliding mode. In Chang 1991, an *integral sliding condition* has been proposed using the same system order, however this approach has some limitation. Alternatively, we propose an integral VSS control by augmenting the order of a system model then the controller is designed in a unified manner as in the normal case without integral. It can solve the problem of pseudo-SMC in Chern *et al.* 1991 and cope with limitation in Chang 1991 (Example 3.9).

3.4. Chapter 4: A New Robust Sliding Mode Controller-Observer Design

In this chapter, a new robust sliding mode controller (SMC) design is fully presented in a unified manner for both discontinuous SMC (VSS) and continuous SMC control functions under parametric uncertainty and external disturbance. The control function is partitioned into 3 components: equivalent control, reaching control, and perturbation control.

For the robust discontinuous SMC, the design proposed in Chapter 3 is extended to deal with uncertain systems. This approach can be directly applied for uncertain MIMO nonlinear systems and it may be the simplest in the current literature (Fu 1992, Chapter 7).

In the proposed linear SMC design, the control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem in the standard VSS design. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework. For example, the sliding margin in the literature can be shown to function as the reaching eigenvalue, the system closed-loop eigenvalues are composed of the sliding margin (reaching eigenvalue) and hyperplane eigenvalues.

A SMC is based on a state-space model, an I-action may be required to eliminate steady-state errors. We will prove that the integral VSS control (discontinuous SMC) in Theorem 3.4 can be still applied to both robust discontinuous and linear continuous SMC controls in this chapter.

Since the principal operating mode of a VSS control is the sliding mode, the VSS control can be seen as a discontinuous subset of the SMC. In this chapter, the state-space control can be seen as a deterministic subset of the SMC.

SMC is a state-space control approach, it requires an observer to estimate unavailable states. In Bondarev *et al.* 1985, a linear Luenberger observer has been used as an observer of a VSS controller for deterministic linear systems (no uncertainties). In Walcott *et al.* 1987 and Yaz *et al.* 1993, a Lyapunov sliding condition has been used to design an observer for a class of systems under matched uncertainty restricted to a certain system structure. In Slotine *et al.* 1987, a sliding patch condition has been used to have a region of direct attraction where uncertainty is not fully tackled. In fact, in Walcott *et al.* 1987, Slotine *et al.* 1987 and Yaz *et al.* 1993, to cope with uncertainty, a Lyapunov sliding condition has been employed to include a switching component into a linear Luenberger observer where a linearized model is used for a nonlinear system. In Edwards *et al.* 1995, an observer has been implemented using output feedback technique.

In this chapter, we propose a novel robust sliding mode observer design for a wide class of systems under both matched and unmatched uncertainty. This design is a development of the proposed robust linear sliding mode controller design.

To complete the work in Drazenovic 1969, we present necessary and sufficient invariance conditions. They are *valid for both discontinuous and continuous SMC.*

3.5. Chapter 5: A New Robust Discrete-Time Sliding-Mode Controller-Observer Design

In this chapter, a new robust discrete-time sliding mode controller (SMC) and observer design is fully presented for both discontinuous SMC (VSS) and continuous SMC control functions, it can be seen as a discrete-time extension of the previous chapter. A high sampling rate is proved to be a necessary condition for a discrete-time SMC, under no uncertainty, this condition reduces to the stability criterion of unit circle in the Z-domain.

Based on the work in Pieper *et al* 1992 for matched uncertain dynamical systems, an alternative design is presented without the constraint stated therein. On the basis of the new robust sliding mode controller design in the previous chapter, a new robust discrete-time sliding mode controller design is developed to deal with uncertain systems. This design thus inherits all features of the previous continuous-time design (reaching, sliding, hyperplane and closed-loop eigenvalues; robust SMC and integral SMC, robust sliding-mode observer, etc.)

3.6. Chapter 6: Robust Sliding-Mode Fuzzy Controller Design

Since the invention of the first fuzzy controller by Mamdani in 1974, fuzzy controllers have been found successfully in numerous industrial applications such as cement-kiln process control, automatic train operation, camcorder autofocusing, crane control, etc. These systems could be classified as slow systems.

In this chapter, a fuzzy rulebase is identified to take 3 forms: soft, sharp and full rulebases. A fuzzification can be normal (linear distribution) or weighted (nonlinear distribution) using triangle or bell membership function. A fuzzy inference can be minimum or product method. A defuzzification can be mean-of-maxima or centroid method. Based on different fuzzy structure (membership function, fuzzification, defuzzification, rulebases, fuzzy inference, etc.), we find the best fuzzy structure (membership function, fuzzification, defuzzification, rulebase, fuzzy inference) applicable to both slow and fast systems.

For the fuzzy control, we have proved that a typical fuzzy rulebase can satisfy the Lyapunov sliding condition so the stability of a fuzzy control is guaranteed by the Lyapunov stability theorem. This is a stability criterion for the fuzzy control theory. We have presented a proposition for a fuzzy control structure applicable to slow and fast systems.

To design a stable sliding-mode fuzzy controller, a fuzzy mechanism is used to minimize a sliding variable s instead of using the sliding condition as in the sliding mode control, so we can obtain the invariance property of the sliding mode. In a typical fuzzy rulebase, it may not be convenient to use more than 2 entries, we can use 1 entry for s and the other for sum of s, and hence a possible steady-state error may be eliminated by this I-action.

In a fuzzy control, using unity gains for error and its change may cause chattering in fast systems (Section 6.4). To adjust these gains means to adapt to system dynamics. Using the sliding-mode control theory, these gains can be determined by a hyperplane and the chattering problem can be solved since the system dynamics are taken into account (Example 6.1).

On the basis of the fuzzy identification in Tanaka *et al.* 1992 and Ishigame *et al.* 1993, we develop a new fuzzy identification scheme which is simpler and more practical. The fuzzy inference will be used to obtain the most potential model from some rough mathematical models from experiments using a proposed practical system identification. Due to the robustness, a rough system model is required rather than an elaborate mathematical model as in a conventional control, a practical system identification is presented for this purpose. A fuzzy model by the proposed scheme can be a solution to the conservative problem.

3.7. Chapter 7: General Sliding-Mode Controller Design for MIMO Uncertain Nonlinear Systems

In this chapter, we present a SMC design for SISO nonlinear systems, and also a design for MIMO systems. So far we have only considered the case where the output is the first system state. In this chapter we will consider the case where the output is a nonlinear function of all system states.

For a MIMO SMC, the hierarchical control technique has been used in Utkin 1977 for linear systems. Alternatively, we will use a decoupling technique which is applicable for MIMO nonlinear systems. This technique allows a MIMO can be considered as a collection of SISO subsystems. As consequence, all SISO results developed so far can be applicable, including the SISO nonlinear SMC in this chapter.

For the general case of multivariable nonlinear SMC, in the SMC literature (Fernandez *et al.* 1987; Chen *et al.* 1992), the hyperplane design has been based on the *Input-Output Linearization* technique (Hunt *et al.* 1983, Isidori 1985, Kravaris *et al.* 1986) to transform a nonlinear system into a canonical nonlinear system. By the nature of a hyperplane that it is of reduced-order, we will design the hyperplane via the direct allocation approach in Chapter 2. The controller will be designed in a unified manner as in other cases. The proposed robust design may be the simplest approach in the literature (Fu 1992, Sira-Ramirez 1996). In addition, the design scheme in Proposition 3.3 is proved to be efficient in solving the chattering and steady-state error (Example 7.6).

3.8. Chapter 8: Advanced Sliding-Mode Controller Design: Experimental Results

In this chapter, results from experiments of Ball-Hoop system are presented to validate our anticipations in theory such as infinite gain of the switching function may cause excitation of unmodelled high-frequency; slow-down system responses due to low gain of saturate function; performances of continuous SMC, continuous pseudo-SMC (TanH SMC) and sliding-mode fuzzy control.

Advanced Sliding-Mode Controller Design: Introduction

Chapter 3

A New Variable-Structure System Controller Designs

3.1. INTRODUCTION

In this chapter, a VSS controller design will be fully developed in a unified manner that is extendable into robust control (Chapter 4) and MIMO nonlinear system (Chapter 7) where a VSS control is a discontinuous sliding mode control (SMC). Based on the concept of sliding-eigenvalues introduced in Chapter 2, we propose a stability criterion which is much simpler than the current approach. By the sliding condition Eq.(1.4), a control function is first computed in a standard form as in Utkin 1977. We then propose a technique to convert this form into a compact form to facilitate controller implementations. The sliding condition is a necessary condition to guarantee the sliding mode. We propose a sufficient condition as a SMC design rule that the reaching mode must terminate in a finite time so that the sliding mode can exist, because the invariance property is with the sliding mode only. A new reaching control design is proposed to include system dynamics for efficient SMC designs, this is not the case in the current literature.

Alternative to the chattering problem analysis in Slotine *et al* 1983, we propose an analysis that allows a unified approach for both the saturate (Slotine *et al.* 1983) and unitvector functions (Ryan *et al.* 1984, Spurgeon 1991). In addition, the hyperbolic tangent function is proposed to cope with limitations of saturate and unitvector functions. Using these sliding functions above (saturate, unitvector and hyperbolic tangent functions) may cause a steady-state error, a design scheme is proposed to solve this problem (Example 7.6).

As a VSS control is based on a state-space model, an I-action may be required to eliminate a steady-state error. In Chern *et al.* 1991, an integral of error has been used, however it is a pseudo-SMC because the larger controller gain the closer the sliding mode. In Chang 1991, an *integral sliding condition* has been proposed using the same system order, however this approach has some limitation with steady-state error. Alternatively, we propose an integral VSS control by augmenting the order of a system model then the controller is designed in a unified manner as in the normal case without integral. It can solve the problem of pseudo-SMC in Chern *et al.* 1991 and cope with limitation in Chang 1991 (Example 3.11).

3.2. SMC STABILITY CRITERION

To our best knowledge, in the VSS literature, there has been a little attention paid to the stability problem, only 2 works in Itkis 1976 and Utkin 1977 where a stable sliding mode implies a stable closed-loop system. These two works are exactly the same approach to deal with this problem, they are based on a numerous complicated theorems and hence it is not convenient to apply for a practical stability test. Also in those 2 works above, for a controller design, it is based on (n–1) system states for a n-ordered system so there is always one constraint while the hyperplane still does include all the n-states, so there is no point to be constrained.

Alternatively, we analyze this stability problem based on the mechanism of the SMC, then we propose a much simpler stability criterion for practical test for SMC.

Theorem 3.1: SMC Stability Criterion

Consider a SISO linear system

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u$$

where

$$\mathbf{x}, \mathbf{B} \in \Re^{n \times 1},\ \mathbf{A} \in \Re^{n \times n},\ u \in \Re$$

if a control function u is designed to satisfy the sliding condition

$$s.\dot{s} < 0 \tag{3.1}$$

and if there exists a stable sliding mode

$$\underline{\underline{\boldsymbol{\lambda}_S = \text{eig}\left[\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA}\right] - \{0\}}} \subset \Re^{1 \times (n-1)}_{(-)} \tag{3.2}$$

then the system is stable

Proof:

We have

$$s.\dot{s} < 0 \Rightarrow \frac{d}{dt}\left(s^2\right) < 0 \Rightarrow s \to 0$$

since s^2 must monotonically reduces to zero as $s^2 > 0$

The sliding mode is characterized by the sliding-eigenvalues determined by Eq.(2.9), if these eigenvalues are Hurwitz, then $\tilde{\mathbf{x}}_1$ converges to *zero* by Eq.(2.14), as does $\tilde{\mathbf{x}}_2$ by Eq.(2.7). By the similarity transformation in Eq.(2.2), $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{bmatrix}^T$ converges to *zero*: the system is stable.

Q.E.D.

Remark 3.1: Stability Test

- All that a SMC has to do is to satisfy the sliding condition $s.\dot{s} < 0$. How negative this value may take is based on the SMC design rule in Proposition 3.1 in Section 3.4.
- This stability test is particularly useful and frequently employed in robust SMC in the next chapter.

3.3. A NEW VSS CONTROLLER DESIGN

In the VSS literature, a control function is either in the standard *if*-form (Slotine *et al.* 1983, White *et al* 1984, Sivaramakrishnan *et al* 1984, Panicker *et al* 1985, Baley *et al.* 1987, DeCarlo *et al.* 1988, Hikita 1988, Hong *et al* 1989) or in the compact form (Young 1978, Ryan 1983, Ambrosino *et al.* 1984, Fernandez *et al.* 1987, DeCarlo *et al.* 1988, Lee *et al* 1991). The derivation of the compact form has been either too complicated to be applied (Ryan 1983) or not clear how to derive (Young 1978, Ambrosino *et al.* 1984, Lee *et al* 1991). In Fernandez *et al.* 1987, the derivation has been based on the input/output linearization technique for nonlinear systems without perturbation. In DeCarlo *et al.* 1988, the derivation is valid only for linear system without perturbation.

The standard *if*-form in Utkin 1977 is a convenient form to write a control function that satisfies the sliding condition. We first propose the following lemma to simplify this standard *if*-form of a VSS control function.

Lemma 3.1: Compact Control Function

For a switching control function of the following standard *if*-form

$$u = \sum_{i=1}^{n} \Psi_i x_i, \qquad \Psi_i = \begin{cases} \psi_i^+, & \text{if } \sigma . x_i > 0 \\ \psi_i^-, & \text{if } \sigma . x_i < 0 \end{cases} \tag{3.3}$$

where σ can be any scalar function, then the *if*-form control function above becomes the compact form as follows

$$u = \sum_{i=1}^{n} \left[\frac{\psi_i^+ + \psi_i^-}{2} x_i + \frac{\psi_i^+ - \psi_i^-}{2} |x_i| . \text{sgn}(\sigma) \right]. \tag{3.1}$$

Proof

Let

$$\Psi_i = \psi_{i0} + \begin{Bmatrix} +\psi_{i1}, & \text{if } \sigma x_i > 0 \\ -\psi_{i1}, & \text{if } \sigma x_i < 0 \end{Bmatrix} = \psi_{i0} + \psi_{i1} . \text{sgn}(\sigma x_i)$$

then

$$\left. \begin{array}{l} \psi_{i0} + \psi_{i1} = \psi_i^+ \\ \psi_{i0} - \psi_{i1} = \psi_i^- \end{array} \right\} \Rightarrow \begin{cases} \psi_{i0} = \dfrac{\psi_i^+ + \psi_i^-}{2} \\ \psi_{i1} = \dfrac{\psi_i^+ - \psi_i^-}{2} \end{cases}$$

thus

$$u = \sum_{i=1}^{n} \left[\psi_{i0} + \psi_{i1} . \text{sgn}(\sigma x_i) \right] . x_i = \sum_{i=1}^{n} \left[\psi_{i0} x_i + \psi_{i1} . \text{sgn}(\sigma x_i) . x_i \right] = \sum_{i=1}^{n} \left[\psi_{i0} x_i + \psi_{i1} |x_i| . \text{sgn}(\sigma) \right].$$

Q.E.D.

Remark 3.2: Simplification of Control Function By Hyperplane Normalization

It is usual that $\sigma = s\mathbf{HB}$, and $\sigma = s$ if the hyperplane is normalized (Section 2.5).

For a switching VSS controller design, we propose the following theorem

Theorem 3.2: Switching VSS Controller Design

For a linear system

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u$$

with a hyperplane

$$s = \mathbf{Hx}$$

then a VSS control function can be combined as

$$u = u_e + u_r \tag{3.4}$$

where

- equivalent control

$$u_e = -\mathbf{K}_e \mathbf{x}, \qquad \mathbf{K}_e = (\mathbf{HB})^{-1} \mathbf{HA} \tag{3.4.a}$$

- reaching control

$$u_r = -(\mathbf{HB})^{-1}\mathbf{K}_r.|\mathbf{x}|.\mathrm{sgn}(s), \quad \mathbf{K}_r = [\delta_1 \quad \cdots \quad \delta_n] \tag{3.4.b}$$

with

$$\mathbf{x},\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{A} \in \Re^{n\times n}, \quad u,s \in \Re, \quad \delta_i \in \Re_{(+)}: \textit{sliding margin.}$$

Proof

We will check the sliding condition, from Eq.(1.8) we have

$$\dot{s} = \mathbf{HB}.(u - u_{eq})$$

where

$$u_{eq} = -(\mathbf{HB})^{-1}\mathbf{HAx} = \sum_{i=1}^{n}\alpha_i x_i, \text{ where } \alpha_i\text{: determined by the system parameters}$$

and

$$u = \sum_{i=1}^{n}\psi_i x_i, \text{ where } k_i\text{: to be determined}$$

thus

$$s\dot{s} = s\mathbf{HB}(u - u_{eq}) = s\mathbf{HB}\sum_{i=1}^{n}(\psi_i - \alpha_i)x_i$$

To satisfy the sliding condition, we obtain

$$s\dot{s} < 0 \Rightarrow \psi_i \begin{cases} < \alpha_i, & \text{if } (s\mathbf{HB})x_i > 0 \\ > \alpha_i, & \text{if } (s\mathbf{HB})x_i < 0 \end{cases} \Rightarrow \psi_i = \begin{cases} \alpha_i - \bar{\delta}_i, & \text{if } (s\mathbf{HB})x_i > 0 \\ \alpha_i + \bar{\delta}_i, & \text{if } (s\mathbf{HB})x_i < 0 \end{cases}, \quad \delta_i > 0$$

by Lemma 3.1, we have

$$\psi_i x_i = \alpha_i x_i - \bar{\delta}_i.|x_i|.\mathrm{sgn}(s\mathbf{HB})$$

then

$$u = \sum_{i=1}^{n}\left\{\alpha_i x_i - \bar{\delta}_i|x_i|.\mathrm{sgn}(s\mathbf{HB})\right\} = -(\mathbf{HB})^{-1}\mathbf{HAx} - \left(\sum_{i=1}^{n}\bar{\delta}_i|x_i|\right).\mathrm{sgn}(s\mathbf{HB}) = -(\mathbf{HB})^{-1}\left[\mathbf{HAx} + \left(\sum_{i=1}^{n}\delta_i|x_i|\right).\mathrm{sgn}(s)\right]$$

where $\delta_i = \bar{\delta}_i.|\mathbf{HB}| > 0$: for the sake of simplicity. **Q.E.D.**

To design a VSS controller using Theorem 3.1, the reaching control constant vector $\mathbf{K}_r$, needs to be determined. For an *n*-order system, there are *n* design parameters δ_i's to be determined. The following corollaries will propose designs using only 1 design parameter δ.

Corollary 3.1: Conventional Reaching Controller Design

The reaching control in Theorem 3.1 can be determined with

$$\mathbf{K}_r = [\delta \quad \cdots \quad \delta] \tag{3.4.c}$$

where $\delta > 0$ is a sliding margin.

Proof

As in the literature, we can choose $\delta_i = \delta$ since the necessary and sufficient condition $\delta_i > 0$ in Theorem 3.1.

Q.E.D.

In the previous approach, the system dynamics are not included in the reaching mode whose dynamics totally depend on the system dynamics as the same sliding margin is used for all system states in all systems (slow and fast systems). The following corollary proposes a more efficient design where the system dynamics are included in the reaching control (Example 3.5)

Corollary 3.2: New Reaching Controller Design

The reaching control in Theorem 3.1 can be determined with

$$\mathbf{K}_r = \delta.|\mathbf{H}| \tag{3.4.d}$$

where $\delta > 0$ is a sliding margin.

Proof

All elements of **H** are associated with all system states, and **H** is designed based on the system matrices **A** and **B.** To included the system dynamics, choosing $\delta_i = \delta.|h_i|$ satisfies the necessary and sufficient condition of $\delta_i > 0$ mentioned in Theorem 3.1 since h_i's are coefficient of the system differential equation.

Q.E.D.

Remark 3.3: Sliding Mode as Fast Mode for Invariance

The sliding margin δ determines the reaching mode, it may be high for the required faster reaching mode as opposed to the sliding mode. So in the literature, the reaching mode is also called the *fast mode,* and the sliding mode called the *slow mode*. By the nature of VSS design that the invariance property is in the sliding mode, therefore the longer is the sliding time (Section 3.4), the more robust is the controller.

3.4. PROPOSITION 3.1: SMC DESIGN RULE

From Remark 3.3, for a proper reconciliation between the reaching mode and the sliding mode, we propose the following design rule applicable to all systems (*linear* or *nonlinear, SISO* or *MIMO*) and all approaches (*continuous* or *discontinuous SMC* or *continuous pseudo-SMC, continuous-time* or *discrete-time SMC, conventional* or *integral SMC,* and *sliding-mode fuzzy control*)

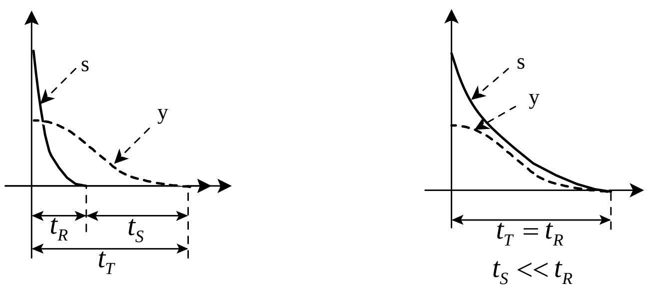

Fig. 3.1: SMC Design Rule

where

s: sliding variable from Eq.(1.2)

y: system output

$t_T = t_R + t_S$, t_T: total time, t_R: reaching time $(s.\dot{s} < 0)$, t_S: sliding time $(s = 0)$

In the figure above, there is a sliding mode in the first case, but practically not in the second.

Response Time = Reaching Time(higher δ→lower t_R) + Sliding Time(higher p_1→lower t_S).

- If $(t_S \approx 0 : t_S << t_R)$, then increasing p_1 does not make the response faster but does *increase the control effort only*!
- Otherwise, if $(t_R \approx 0 : t_R << t_S)$, then increasing δ does not make the response faster but does *increase the control effort only*!

So we propose the following design rule that will be effective in both simulation and experiment.

SMC Design Rule: Ensure $t_S < t_R$, but *not* $(t_S \approx 0 : t_S << t_R)$ and *not* $(t_R \approx 0 : t_R << t_S)$.

3.5. SOLUTION TO CHATTERING PROBLEM

In the VSS literature, there are 2 approaches to eliminate the chattering by using the saturate function (Slotine *et al* 1983) and the unitvector function (Ambrosino *et al.* 1984, Spurgeon 1991). The analysis of the chattering in Slotine *et al.* 1983 raises 2 questions. Firstly why this chattering still exists even in a simulation where the switching controller is practically free from the imperfection. Secondly, based on this analysis, it is not clear how to choose the width of the boundary layer. The elimination of chattering is implied by using the unitvector function in Ambrosino *et al.* 1984, Spurgeon 1991 because this function is continuous. The derivation of this function is either unavailable (Ambrosino *et al.* 1984) or too mathematically complicated to be applied (Ryan *et al.* 1984, Spurgeon 1991), and it is not clear how to choose δ which is the width of the boundary layer in the sense in Slotine *et al.* 1983.

Because of the *infinite gain* of a relay represented by the *sign* function of a typical VSS control function as in Eq.(3.5), normally there is a problem of *chattering*. Our approach raises no questions as mentioned above in Slotine *et al.* 1983, Ambrosino *et al.* 1984, Spurgeon 1991. To eliminate this problem, we may use either a saturation method (this section) or continuous VSS (Chapter 4).

We analyze this problem from another viewpoint: an analysis that allows a unified design for both saturate and unitvector functions. In addition, we propose the hyperbolic tangent function since it is a standard mathematical function and it may outperform the saturate and unitvector functions.

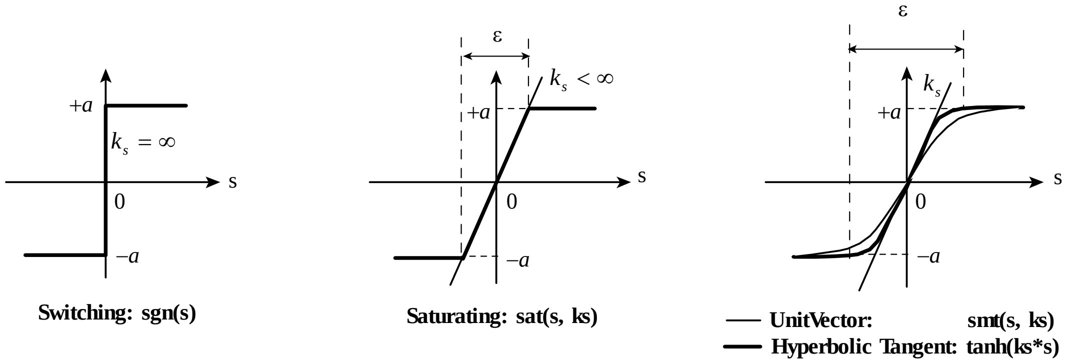

Fig. 3.2: SMC Sliding Functions

3.5.1. Switching Function

A switching function is defined as

$$\operatorname{sgn}(s)=\begin{cases}+a, & \text{if } s>0\\ 0, & \text{if } s=0\\ -a, & \text{if } s<0\end{cases} \tag{3.5}$$

3.5.2. Saturate Function

The sliding gain k_S is reduced from k_S=∞ as it is in a switching function, so a saturate function can be defined as

$$\operatorname{sat}(s,k_s)=\begin{cases}k_s.s, & \text{if } |s|\le \dfrac{1}{k_s}\\ \operatorname{sgn}(s), & \text{if } |s|>\dfrac{1}{k_s}\end{cases} \tag{3.6}$$

3.5.3. Unitvector Function

Alternatively, a switching function in Eq.(3.5) can be defined as:

$$\operatorname{sgn}(s)=\frac{s}{|s|}$$

Hence, to avoid an infinite gain as above, a *unitvector* can defined as follows

$$\operatorname{smt}(s,k_s)=\frac{s}{\dfrac{1}{k_s}+|s|} \qquad \textit{smt} \text{ for } \textit{smooth} \text{ since its appearance} \tag{3.7}$$

since

$$\frac{s}{|s|}\rightarrow\frac{s}{r+|s|}$$

where

$$\begin{cases}|s|>>r: & \dfrac{s}{r+|s|}\approx\dfrac{s}{|s|}=sgn(s)\\ |s|<<r: & \dfrac{s}{r+|s|}\approx\dfrac{1}{r}s\end{cases}$$

with

$$k_s: \text{ the gain reduced from } \infty, \text{ because } \quad \frac{s}{|s|}=\frac{s}{\dfrac{1}{\infty}+|s|}$$

3.5.4. Hyperbolic Tangent Function (TanH)

To avoid an infinite gain, a *hyperbolic tangent* function can be used instead

$$\tanh(k_s s) \tag{3.8}$$

and

$$\left.\frac{d\tanh(k_s s)}{ds}\right|_{s=0}=k_s$$

Remark 3.4: Pseudo-SMC with Boundary Layer

To satisfy the sliding condition, a VSS control function takes the form of a switching function that is defined by Eq.(3.5), so there are only 3 valid values: $-a, 0, +a$.

For a sliding function, outside the band of ε are the valid values $-a, +a$, so the sliding condition is satisfied. Except the origin, inside the band of ε are invalid values, so the *sliding condition is violated in this band*. There is no sliding mode in this band, therefore a VSS control using a sliding function is called a *pseudo-SMC*.

3.5.5. Performance of Sliding Functions (Saturate, Unitvector and TanH Functions)

(1) The switching function has infinite gain while the sliding functions have finite gains. The less gain is, the less chattering is. However, the more possible steady-state error and a slow-down response may results.

(2) As it will be shown in the experimental result at the final chapter, the real danger of the chattering problem is that it excites high-frequency unmodelled plant dynamics. So there is always an *upper bound for the sliding margin*, this bound is the lowest for a switching function and the highest for a unitvector function. So, by Proposition 3.1 in Section 3.4 above, for a switching function, it is possible that the upper bound is low enough to slow down the system response even with the fast eigenvalue.

(3) For a saturate function, the gain is constant (k_s of the saturate function in the figure above) within the boundary layer. For unitvector and hyperbolic functions, the gain decreases when the value of the unitvector function increases ($k_s^* < k_s$ of the unitvector function in the figure above), thus the less possible for an oscillation (chattering) to occur

- This is *a disadvantage of the unitvector and hyperbolic tangent over the saturate function*, because the more possible for a steady-state error to occur.
- On the other hand, this is an *advantage of the unitvector and hyperbolic tangent over the saturate function.* The upper bound of the saturate function is lower than that of a unitvector function. For the saturate function, this bound may be low enough to slow down the system response even with the faster eigenvalue.

Since the gains of all sliding functions above are lower than that of the sign function (infinite gain), a steady-state error may occur. In addition, the chattering is excited by a large error. The following proposition is used to solve this problem

Proposition 3.2: Application of Sliding Function

To eliminate the chattering problem, the design procedure is exactly the same, we simply to replace a *switching function* by a *saturate*, a *unitvector* or a *TanH function*. The gain may be chosen just low enough to eliminate the chattering. The lower is the gain, the wider is the boundary layer in which a pseudo-VSS control exists (Remark 3.4).

The following proposition eliminates the steady-state error arised in using the sliding function to eliminate the chattering (Example 7.6).

Proposition 3.3: Solution to Problem of Chattering and Steady-State Error

```
if error > err_tol
   u is based on the saturate, unitvector or hyperbolic function
else
   u is based on the sign function
```

where `err_tol` is chosen small enough to eliminate the chattering and error.

3.6. INTEGRAL VSS CONTROLLER DESIGN

A VSS control is based on a state-space model, an I-action may be required to eliminate a steady-state error. In Chern *et al.* 1991, an integral of error has been used, however it is a pseudo-SMC because the larger controller gain the closer the sliding mode. In Chang 1991, an integral sliding condition has been proposed using the same system order, however this approach can be applied to some systems but not all (Example 3.11). Alternatively, we propose an integral VSS control by augmenting the order of a system model then the controller is designed in a unified manner as in the case without integral. It can solve the problem of pseudo-SMC in Chern *et al.* 1991 and cope with limitation in Chang 1991.

Based on the integral sliding condition in Chang 1991, we have the following theorem to design an integral SMC.

Theorem 3.3: Integral VSS Controller Design by Integral Sliding Condition

Consider the following system

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3.9}$$

and a hyperplane

$$s = \mathbf{H}\mathbf{x}$$

if $\tilde{u}$ is a SMC of the system in Eq.(3.9), then there exists a constant δ_i to determine an integral SMC as

$$u = \tilde{u} - (\mathbf{HB})^{-1}\delta_i \int_0^t s.dt, \qquad \delta_i > 0 \tag{3.10}$$

where δ_i is a positive constant as an integral sliding margin and can be determined by the SMC design rule in Proposition 3.1.

Proof

A Lyapunov function can be defined as

$$V = \tfrac{1}{2}s^2 + \tfrac{1}{2}\delta_i\left(\int_0^t s\,dt\right)^2 \tag{3.11}$$

so

$$\dot{V} = s\left(\dot{s} + \delta_i \int_0^t s\,dt\right)$$

then from Eq.(3.9), we have

$$\dot{V} = s.\left(\mathbf{HAx} + \mathbf{HB}u + \delta_i \int_0^t s\ dt\right)$$

by Eq.(3.10), we have

$$\dot{V} = s.\left(\mathbf{HAx} + \mathbf{HB}\tilde{u}\right)$$

thus

$$\dot{V} < 0 \tag{3.12}$$

since $\tilde{u}$ is a SMC of Eq.(3.9).

Q.E.D.

Remark 3.5: Applicability of SMC Design Rule to Integral VSS

In Theorem 3.3, the control $\tilde{u}$ is designed normally as a SMC and an integral of sliding variable s is included with an integral sliding margin δ_i determined by the SMC design rule in Proposition 3.1.

Alternatively, we have the following theorem to design an integral SMC where the system model is augmented and an integral of error is used instead of sliding variable to improve an I-action. The advantage of the design in Theorem 3.3 is simple since the system model is not used in the I-action, however the following design can cope with its limitation (Example 3.10 and 3.11) because all system states are integrated in Theorem 3.3 via the sliding variable but only the error of output in the following theorem.

Theorem 3.4: Integral VSS Controller Design by Augmented System

Consider a system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \\ y = \mathbf{Cx} \end{cases} \tag{3.13}$$

and its augmented-order system with a reference input of r

$$\mathbf{x}_i = \mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i u_i \tag{3.14}$$

where

$$\mathbf{x}_i = \begin{bmatrix} \int_0^t (y-r)\,dt \\ \mathbf{x} \end{bmatrix}, \quad \mathbf{A}_i = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{0}_{n\times 1} & \mathbf{A} \end{bmatrix}, \quad \mathbf{B}_i = \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} \tag{3.14.a}$$

then an integral VSS control with a hyperplane $\mathbf{H}_i$ can be determined by

$$\underline{\underline{u = u_i + \tilde{u}}} \tag{3.15}$$

where

$$\tilde{u} = \left(\mathbf{H}_i \mathbf{B}_i\right)^{-1} h_{i,1}\ r, \quad \mathbf{H}_i = \begin{bmatrix} h_{i,1} & \cdots & \tilde{h}_{i,n+1} \end{bmatrix} \tag{3.15.a}$$

and

u_i is a VSS control for Eq.(3.14) and can be determined by Theorem 3.2 or 3.3.

with

$\mathbf{x}, \mathbf{B} \in \Re^{n\times 1}$, $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{C} \in \Re^{1\times n}$, $u, \tilde{u}, u_i, y, r \in \Re$: sliding margin.

Proof

Let

$$x_0 = \int_0^t (y - r)\, dt \quad \Rightarrow \quad \dot{x}_0 = y - r$$

from Eq.(3.13), we have

$$\dot{\mathbf{x}}_i = \begin{bmatrix} \dot{x}_0 \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} y - r \\ \mathbf{Ax} + \mathbf{B}u \end{bmatrix} = \begin{bmatrix} \mathbf{Cx} \\ \mathbf{Ax} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}u \end{bmatrix} - \begin{bmatrix} r \\ \mathbf{0}_{n\times1} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{0}_{n\times1} & \mathbf{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times1} \end{bmatrix} r$$

or, by Eq.(3.14a)

$$\dot{\mathbf{x}}_i == \mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times1} \end{bmatrix} r \tag{3.16}$$

thus

$$\dot{s}_i = \mathbf{H}_i \dot{\mathbf{x}}_i = \mathbf{H}_i \mathbf{A}_i \mathbf{x}_i + \mathbf{H}_i \mathbf{B}_i u + \mathbf{H}_i . \begin{bmatrix} -1 \\ \mathbf{0}_{n\times1} \end{bmatrix} . r = \mathbf{H}_i \mathbf{A}_i \mathbf{x}_i + \mathbf{H}_i \mathbf{B}_i u - h_{i,1}\, r \tag{3.17}$$

by Eq.(3.15), we have

$$\dot{s}_i = \mathbf{H}_i \mathbf{A}_i \mathbf{x}_i + \mathbf{H}_i \mathbf{B}_i \left(u - \tilde{u} \right) == \mathbf{H}_i \mathbf{A}_i \mathbf{x}_i + \mathbf{H}_i \mathbf{B}_i u_i \tag{3.18}$$

since u_i is a VSS control for Eq.(3.14), the sliding condition $s_i \dot{s}_i \le 0$ is satisfied using Eq.(3.18)

Q.E.D.

3.7. NUMERICAL EXAMPLES

For the purpose of comparison only, our design keeps the control effort in the same order as in the original design, although in some examples it is rather high. Based on the *SMC design rule* in Proposition 3.1 (Section 3.4), our proposed control functions are designed in a unique approach, while the original control functions have been designed in different approaches: equal excursion sliding-mode, non-linear hyperplane.

Hyperplane designs are omitted, they can be found in the previous chapter of the hyperplane design. In fact, we have built a VSS toolbox in MATLAB language and all the following hyperplanes are determined by the corresponding function in that toolbox. It is necessary to emphasize that **H** is normalized such that **HB=I.** By this choice, we not only simplify the design calculation, but also get the unique equation for a hyperplane rather than a different equation by a scaling factor.

All the hyperplane design methods in the previous chapter are used: the eigenvalues allocation method, the optimal sliding mode method, and the direct calculation method. Since the original control function is in a switching form, the corresponding proposed control functions are also in the switching form for the purpose of comparison. To see that the hyperplane design is independent of the controller design, there are different VSS control functions for each hyperplane design method.

For the saturate VSS or unitvector VSS design, it is exactly the same as the switching counterpart with the extra introduction of K_S. It may be chosen just low enough to eliminate the chattering.

The first 4 examples are designed in all methods for an assessment, each starts with the original design from the VSS literature, next follows with new designs for switching VSS then saturate, unitvector and TanH VSS to solve the chattering problem. Each concludes with I/O-State and optimal VSS designs in switching control functions to compare with the original design which is also in switching control function, other designs (saturate, unitvector and TanH VSS) are not presented to avoid a lengthy presentation.

The next 3 examples are used to illustrate some issues such as the efficiency of the new design, the validity of the application of the I/O-state method (Section 2.3.4), the validity of the SMC design rule (Section 3.4) and the problem of a steady-state error when reducing the gain in the saturation approach (Section 3.5). The following example is used to show the efficiency of the proposed integral VSS control. As mentioned in the previous chapter on the hyperplane design, the assessment of all the hyperplane design methods must be made in this chapter where the VSS control function is available. So in this chapter, there are comparisons not only for the hyperplane design methods (eigenvalue allocation method, I/O-state method, optimal method), but also for the controller design methods (switching, saturate, unitvector and TanH functions).

Remark 3.6: Summary of VSS Designs

In the numerical examples below, and in this work generally, hyperplane eigenvalues will be chosen at the same unique value for simplicity except in case of the optimal sliding mode approach. Different multiple values may be attempted to compromise between the response speed and overshoot.

- Hyperplane designs are referred to the previous chapter;
- Sliding margin δ is chosen on the basis of Proposition 3.1;
- New Switching VSS control functions are computed by Theorem 3.2 and Corollary 3.2 where the switching function is given by Eq.(3.5);
- Saturate, Unitvector and TanH VSS control functions are given by Eqs.(3.6) to (3.8), respectively, based on Proposition 3.2;
- Integral VSS control functions can be determined by Theorem 3.3 (Integral VSS) or Theorem 3.4 (Augmented System)

3.7.1. Example 3.1: Canonical System

3.7.1.1. Original Design

Consider a system in White *et al.* 1984

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

with the hyperplane

$$s = \mathbf{H}\mathbf{x}, \qquad \mathbf{H} = [38, \quad 9, \quad 1]$$

and the original control function is given as

$$u = -(\mathbf{K} + \Delta\mathbf{K})\mathbf{x}, \qquad \mathbf{K} = [-120, \quad 0, \quad 0], \qquad \Delta\mathbf{K} = \begin{cases} [+150, \quad +30, \quad +5], & \text{if } sx_i > 0 \\ [-150, \quad -30, \quad -5], & \text{if } sx_i < 0 \end{cases}$$

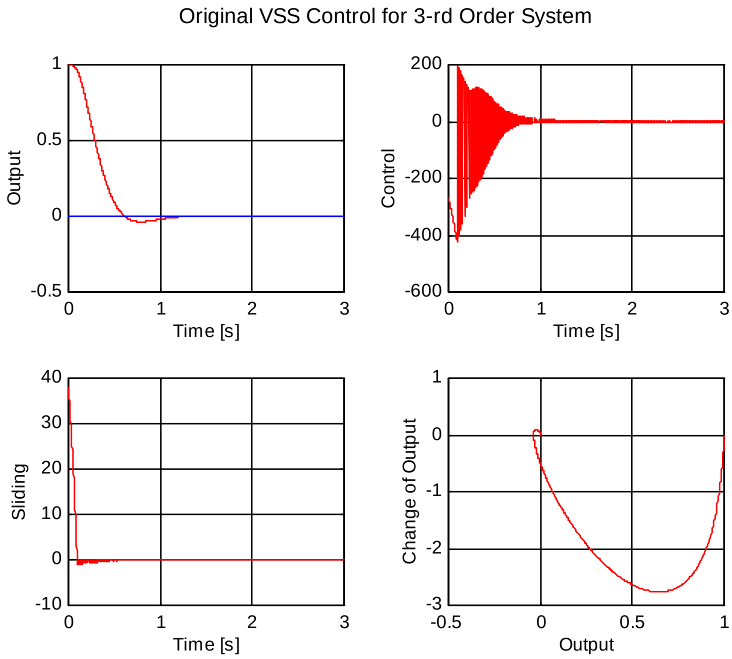

Fig. 3.3: Original VSS Control for Example 3.1.

3.7.1.2. New Designs

Choose $\lambda_H = [-7, \quad -7]$, we have

$$\mathbf{H} = [49, \quad 14, \quad 1]$$

thus

$$\mathbf{K}_e = [-6, \quad 38, \quad 8]$$

Choose $\delta = 2.5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [98 \quad 28 \quad 2]$$

(a) Switching VSS

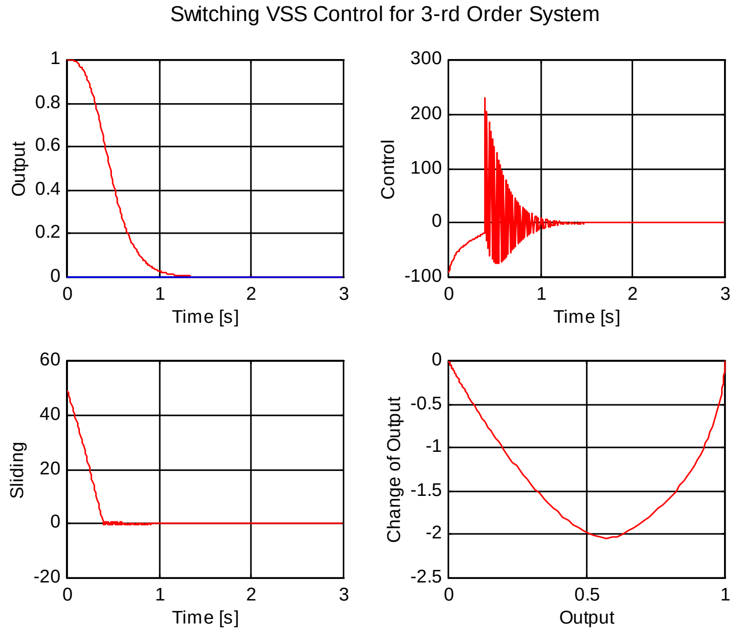

Fig. 3.4: Switching VSS Control for Example 3.1.

(b) Saturate VSS

Choose $k_s = 0.75$, then

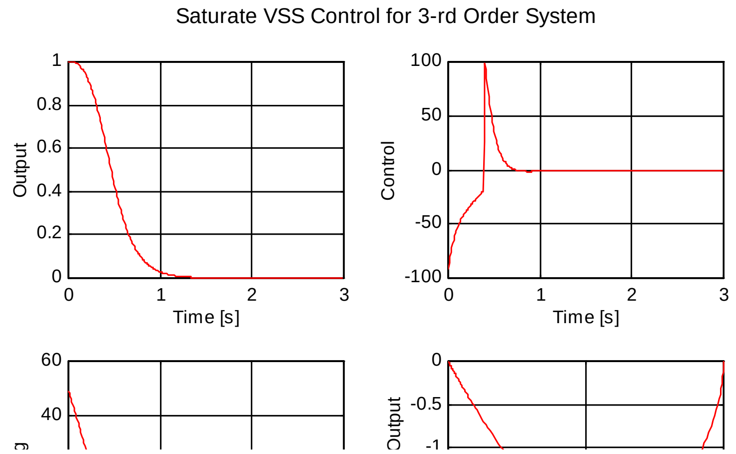

Fig. 3.5: Saturate VSS Control for Example 3.1.

(c) Unitvector VSS

Choose $k_s = 1.75$, then

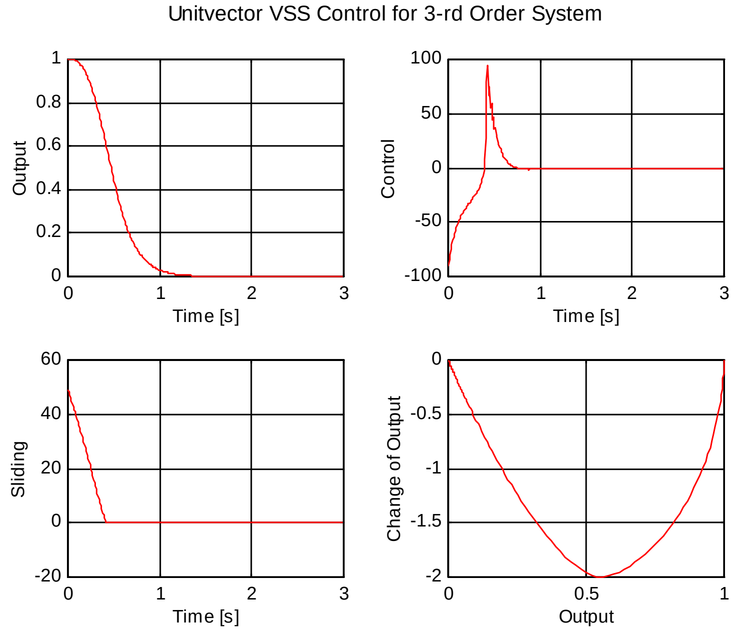

Fig. 3.6: Unitvector VSS Control for Example 3.1.

(d) TanH VSS

Choose $k_s = 1.25$, then

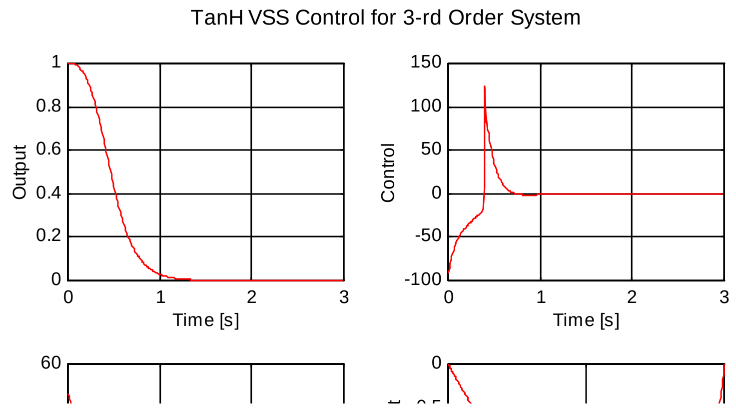

Fig. 3.7: TanH VSS Control for Example 3.1.

(e) Switching VSS by I/O-state of Relative Degree *r*

We have

$$n = 3, \qquad r = 3$$

so the resulting hyperplane by the I/O state method is similar to the above hyperplane due to the canonical system

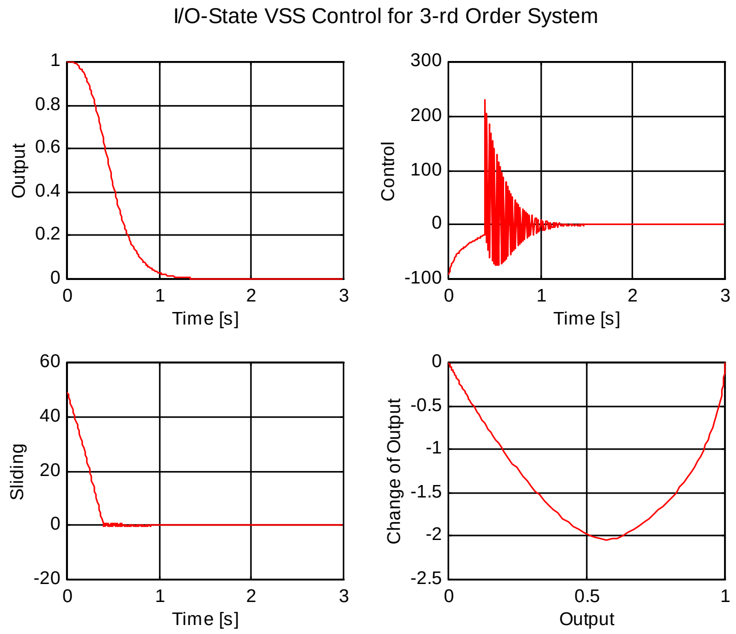

Fig. 3.8: I/O State Switching VSS Control for Example 3.1.

(f) Error-Optimal Sliding Mode

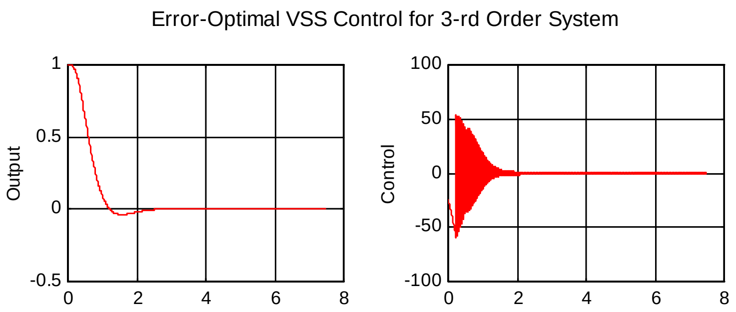

Fig. 3.9: Error Optimal VSS Control for Example 3.1.

Choose $\mathbf{Q} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$, then

$$\mathbf{H} = [10, \quad 4.5826, \quad 1]$$

thus

$$\mathbf{K}_e = [-6, \quad -1, \quad -1.4174]$$

choose $\delta = 3$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [30.0 \quad 13.7477 \quad 3.0]$$

3.7.1.3. Discussion

- I/O-State method: the same since the system is canonical.
- Error-optimization: comparable to the normal approach since the control effort reduces by half but the response time increases by twice.

3.7.2. Example 3.2: Non-canonical System

3.7.2.1. Original Design

Consider a system in Sivaramakrishnan *et al.* 1984

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.05 & 6 & 0 & 0 \\ 0 & -3.333 & 3.333 & 0 \\ -5.208 & 0 & -12.5 & -12.5 \\ 0.6 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 12.5 \\ 0 \end{bmatrix} . u$$

with the hyperplane

$$s = \mathbf{H}\mathbf{x}, \qquad \mathbf{H} = [5.155, \quad 4.385, \quad 1 \quad 16]$$

and the original control function

$$u = -\mathbf{K}.\mathbf{x}, \qquad \mathbf{K} = \begin{cases} [+6, \quad +6, \quad +2, \quad 0], & \text{if } s.x_i > 0 \\ [-6, \quad -6, \quad -2, \quad 0], & \text{if } s.x_i < 0 \end{cases}$$

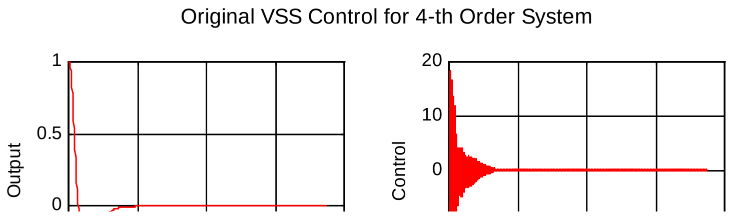

Fig. 3.10: Original VSS Control for Example 3.2.

3.7.2.2. New Designs

Choose $\lambda_H = [-6, \quad -6, \quad -6]$, then

$$\mathbf{H} = [0.4285, \quad 0.3508, \quad 0.0800, \quad 1.4401]$$

thus

$$\mathbf{K}_e = [0.4260, \quad 1.4014, \quad 0.1694, \quad -1]$$

choose $\delta = 10$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [4.2845 \quad 3.5084 \quad 0.80 \quad 14.4014]$$

(a) Switching VSS

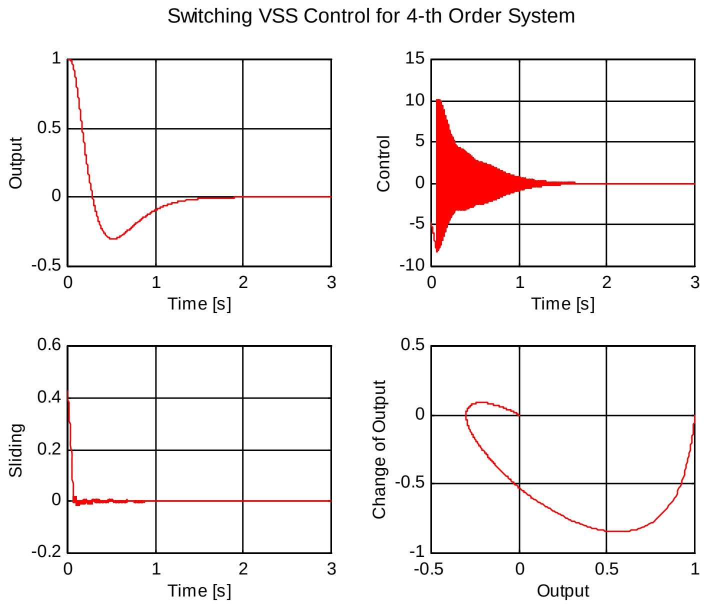

Fig. 3.11: Switching VSS Control for Example 3.2.

(b) Saturate VSS

Choose $k_s = 12$, then

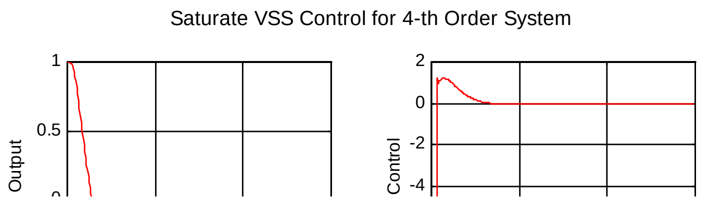

Fig. 3.12: Saturate VSS Control for Example 3.2.

(c) Unitvector VSS

Choose $k_s = 18$, then

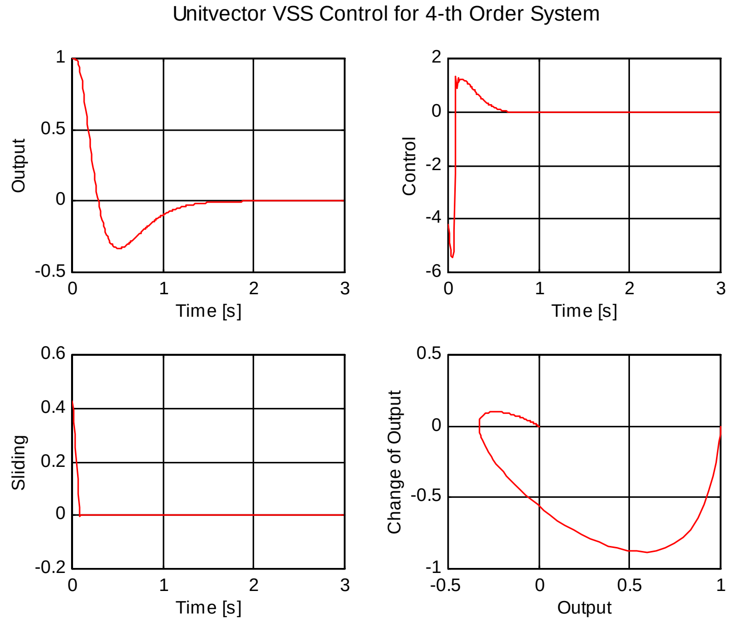

Fig. 3.13: Unitvector VSS Control for Example 3.2.

(d) TanH VSS

Choose $k_s = 20$, then

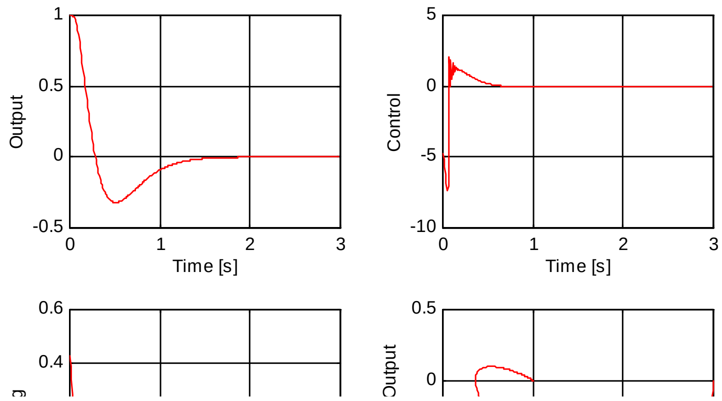

Fig. 3.14: TanH VSS Control for Example 3.2.

(e) Switching VSS by I/O-state of Relative Degree *r*

We have

$$n = 4, \quad r = 3 \quad \Rightarrow \quad \lambda_S = [-6, \ -6, \ 0]\text{: stable, so we continue to design a controller}$$

Choose $\lambda_{H,IO} = [-6, \quad -6]$, then

$$\mathbf{H} = [0.1416, \quad 0.2068, \quad 0.08 \quad 0]$$

thus
$$\mathbf{K}_e = [-0.4237, \quad 0.1604, \quad -0.3106, \quad -1]$$

choose $\delta = 2$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [0.2832 \quad 0.4137 \quad 0.1600 \quad 0]$$

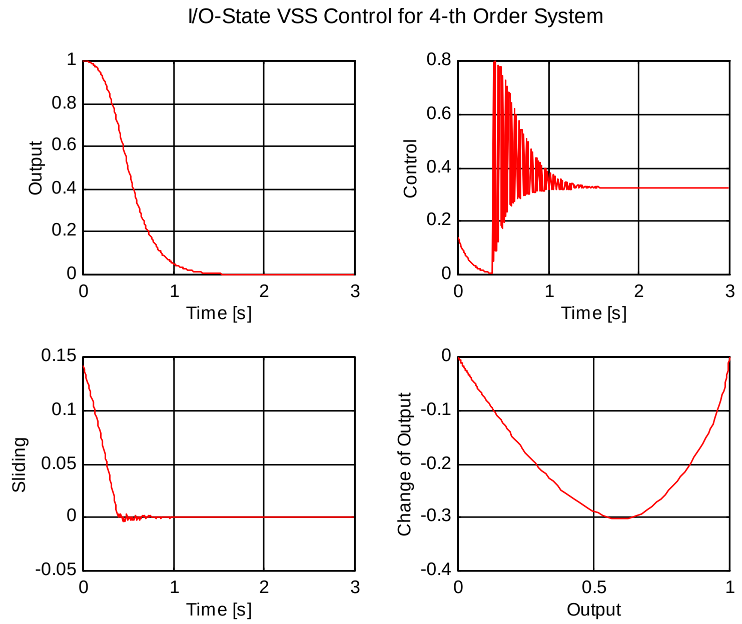

Fig. 3.15: I/O-State VSS Control for Example 3.2.

(f) Error-Optimal Sliding Mode

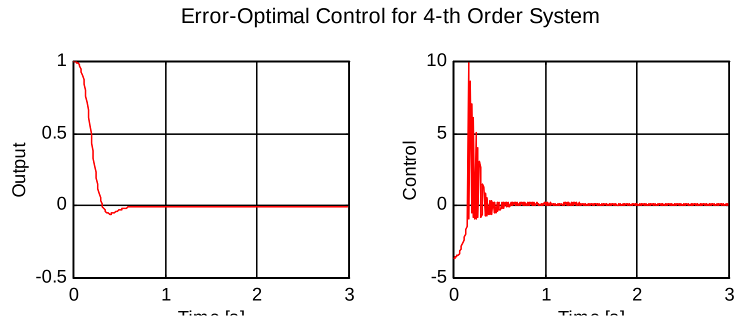

Fig. 3.16: Error-Optimal VSS Control for Example 3.2.

Choose $\mathbf{Q} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$, then

$$\mathbf{H} = [0.8008, \quad 0.4134, \quad 0.08, \quad 0.08]$$

thus
$$\mathbf{K}_e = [-0.4087, \quad 3.4270, \quad 0.3779, \quad -1]$$

choose $\delta = 5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [4.0041 \quad 2.0671 \quad 0.4000 \quad 0.4000]$$

3.7.2.3. Performance compared with the Normal Case

- I/O-State method: same speed, no overshoot, much higher efficiency.
- Error-optimization: faster and smaller overshoot.

3.7.3. Example 3.3: Equal Excursion Sliding Mode

3.7.3.1. Original Design

Consider a system in Hong *et al.* 1989

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.\mathbf{x} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}.u$$

with the hyperplane

$$s = \mathbf{H}\mathbf{x}, \qquad \mathbf{H} = [0.7, \quad 1]$$

and the original control function

$$u = k_1 x_1 - 0.43 x_2, \qquad k_1 = \begin{cases} 0.6, & \text{if} \quad s x_1 > 0 \\ -1.26, & \text{if} \quad s x_1 < 0 \end{cases}$$

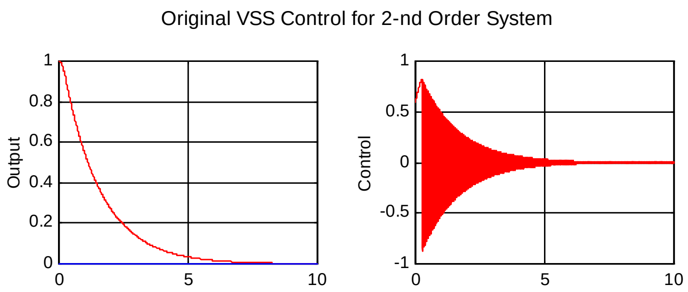

Fig. 3.17: Original VSS Control for Example 3.3.

3.7.3.2. New Designs

Choose $\lambda_H = [-1]$, then

$$\mathbf{H} = [-0.3333, \quad -0.3333]$$

thus

$$\mathbf{K}_e = [0.3333, \quad 0.3333]$$

choose $\delta = 1.5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [0.5, \quad 0.5]$$

(a) Switching VSS

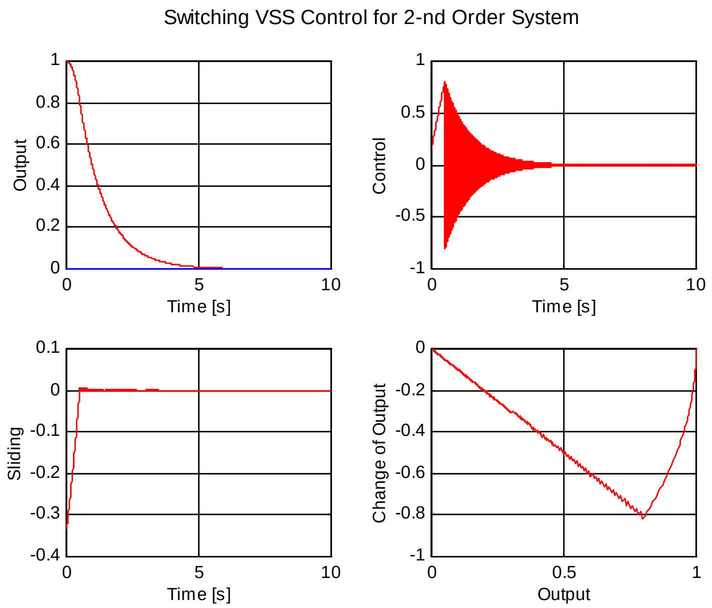

Fig. 3.18: Switching VSS Control for Example 3.3.

(b) Saturate VSS

Choose $k_s = 150$, then

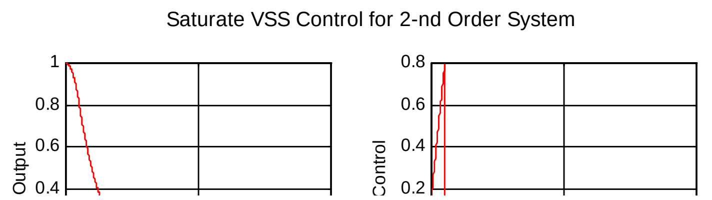

Fig. 3.19: Saturate VSS Control for Example 3.3.

(c) Unitvector VSS

Choose $k_s = 250$, then

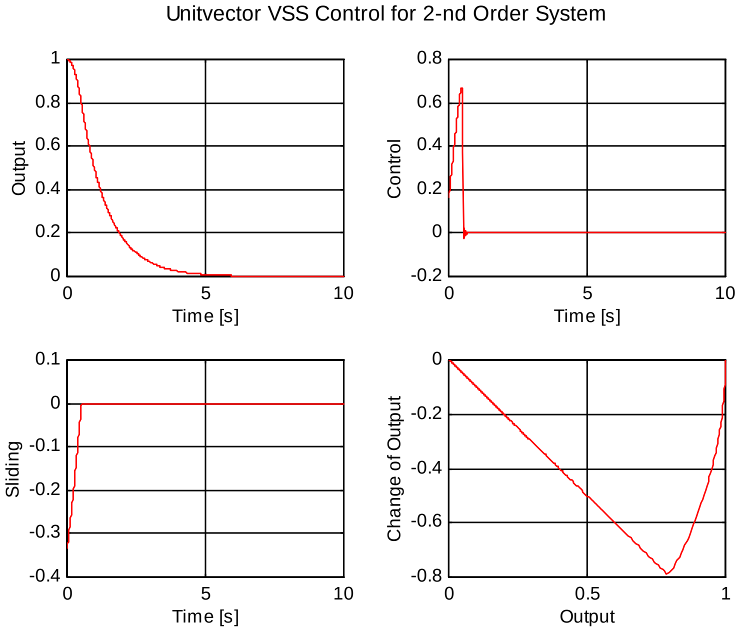

Fig. 3.20: Unitvector VSS Control for Example 3.3.

(d) TanH VSS

Choose $k_s = 200$, then

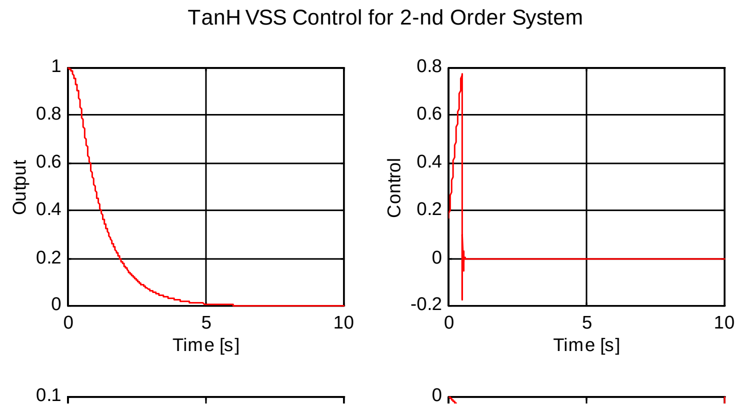

Fig. 3.21: TanH VSS Control for Example 3.3.

(e) Switching VSS by I/O-state of Relative Degree *r*

We have $n = 2, \quad r = 2$

similar to the conventional case due to the canonical system.

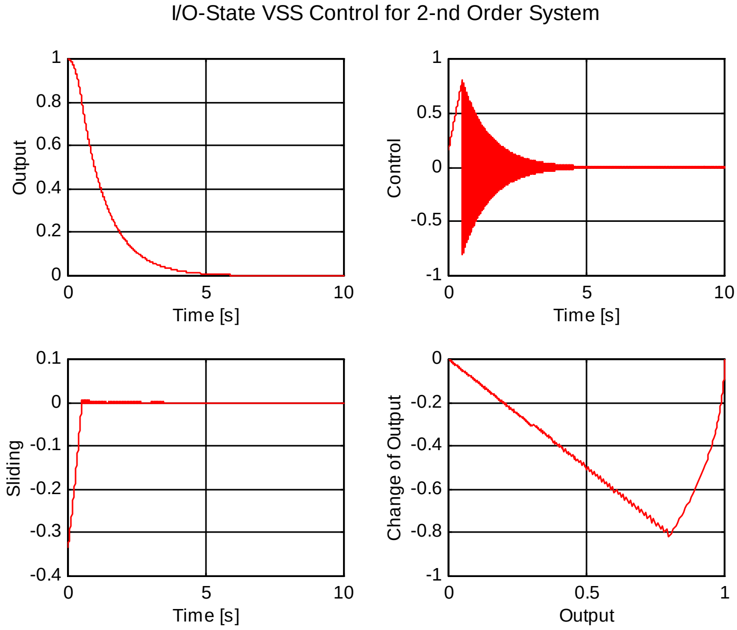

Fig. 3.22: I/O-State VSS Control for Example 3.3.

(f) Error-Optimal Sliding Mode

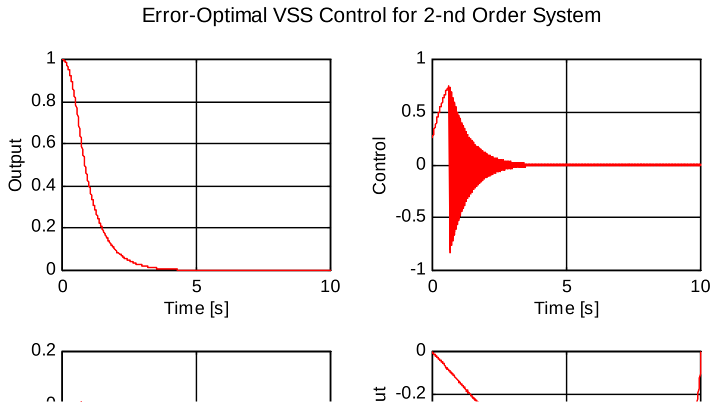

Fig. 3.23: Error-Optimal VSS Control for Example 3.3.

Choose $\mathbf{Q} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$, we have then

$$\mathbf{H} = \begin{bmatrix} -0.4714, & -0.3333 \end{bmatrix}$$

thus
$$\mathbf{K}_e = \begin{bmatrix} 0.333, & 0.1953 \end{bmatrix}$$

choose $\delta = 1.25$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = \begin{bmatrix} 0.5893 & 0.4167 \end{bmatrix}$$

3.7.3.3. Discussion

- I/O-State method: the same since the canonical system
- Error-optimization: comparable to the normal approach.

Remark 3.7: Comparison between Proposed Design and Equal Excursion Sliding-Mode Design

In the original design, the author has proposed the technique of an equal excursion sliding mode and illustrated by the example. We still use our normal design approach and still get the same result of equal excursion!

3.7.4. Example 3.4: Non-linear Hyperplane

3.7.4.1. Original Design

Consider a system in Lee *et al.* 1991

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

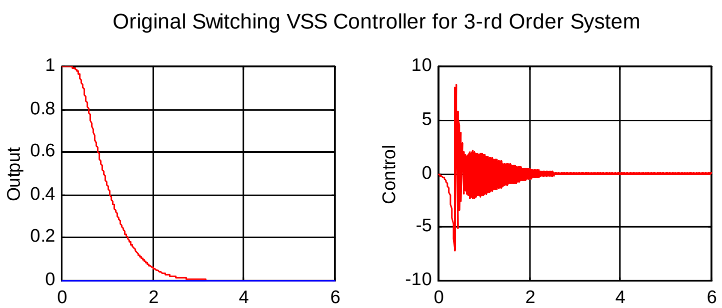

Fig. 3.24: Original VSS Control for Example 3.4.

with the *non-linear* hyperplane

$$s = 9x_1 + 6x_2 + x_3 + 0.33x_1^3 + 1.79x_1^2x_2 - 0.42x_1x_2^2 + 0.48x_2^3$$

and the original control function

$$\begin{cases} u = -\left\{\sum_{i=1}^{3}\phi_i|x_i| + \psi_1|x_1^2x_2| + \psi_2|x_1^2x_3| + \psi_3|x_1x_2^2| + \psi_4|x_1x_2x_3| + \psi_5|x_2^3| + \psi_6|x_2^2x_3|\right\}\mathrm{sgn}(s) \\ \phi = [0.1, \quad 1, \quad 1], \qquad \psi = [0.2, \quad 0.2, \quad 0.5, \quad 0.1, \quad 0.1, \quad 0.2] \end{cases}$$

3.7.4.2. New Designs

Choose $\lambda_H = [-3, \quad -3]$, then

$$\mathbf{H} = [0.9, \quad 0.6, \quad 0.1]$$

thus

$$\mathbf{K}_e = [0, \quad 0.8, \quad 0.6]$$

choose $\delta = 1.5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [1.3500 \quad 0.9000 \quad 0.1500]$$

(a) Switching VSS

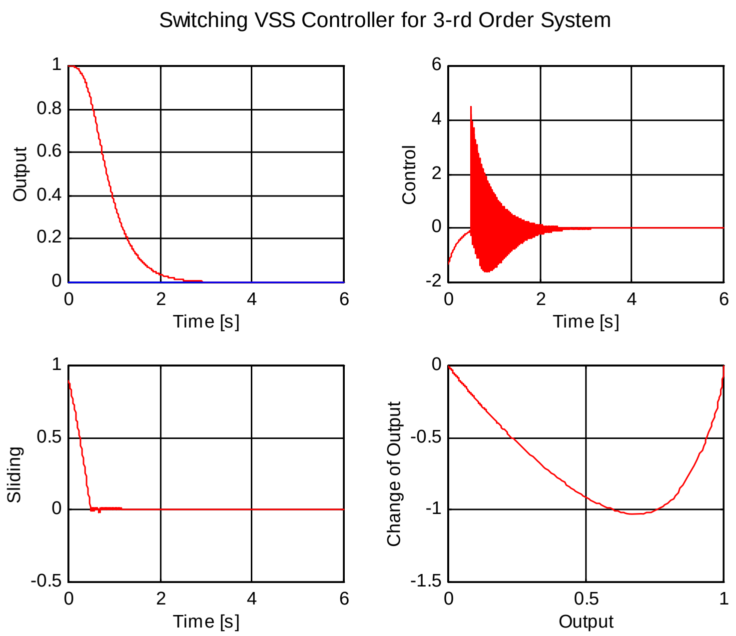

Fig. 3.25: Switching VSS Control for Example 3.4.

(b) Saturate VSS

Choose $k_s = 50$, then

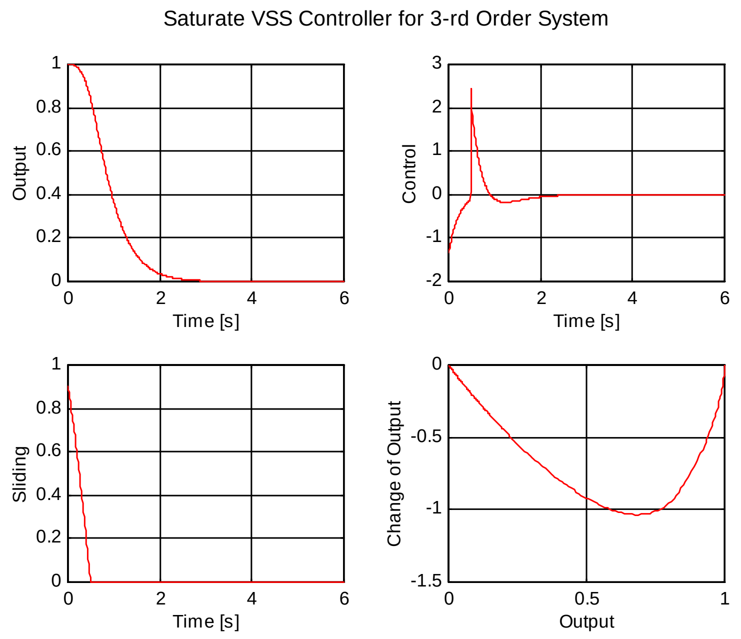

Fig. 3.26: Saturate VSS Control for Example 3.4.

(c) Unitvector VSS

Choose $k_s = 100$, then

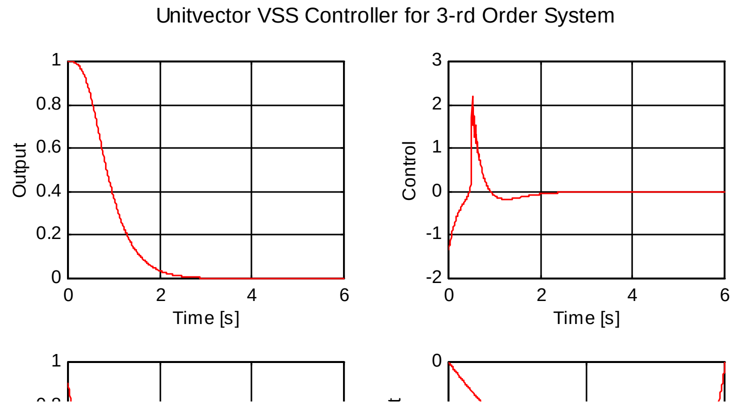

Fig. 3.27: Unitvector VSS Control for Example 3.4.

(d) TanH VSS

Choose $k_s = 75$, then

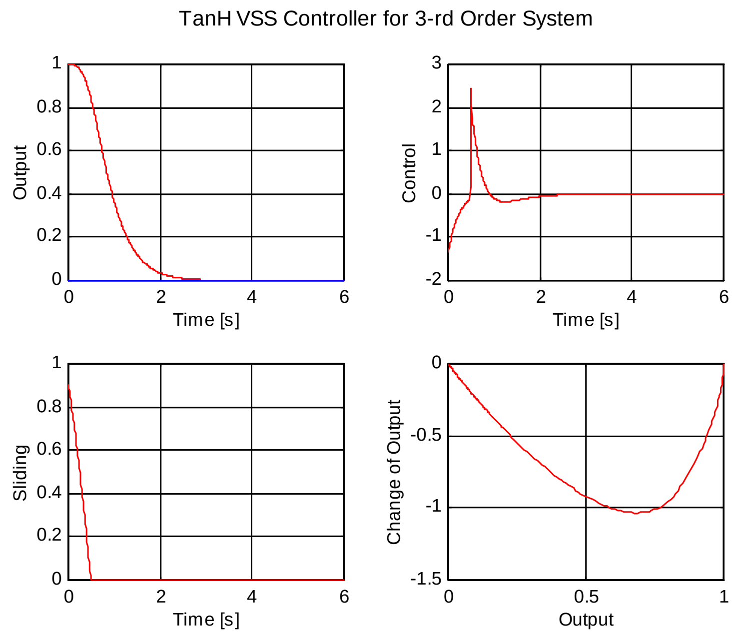

Fig. 3.28: TanH VSS Control for Example 3.4.

(e) Switching VSS by I/O-state of Relative Degree *r*

We have

$$n = 3, \qquad r = 3$$

similar to the conventional case due to the canonical system.

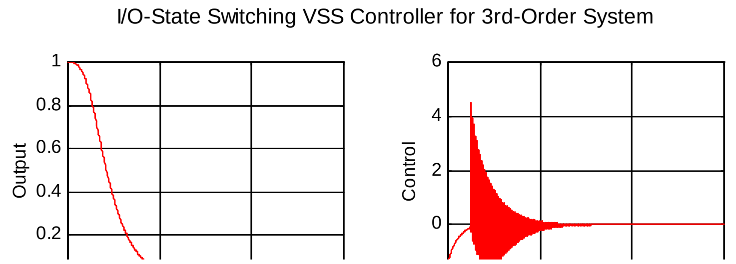

Fig. 3.29: I/O-State VSS Control for Example 3.4.

(f) Error-Optimal Sliding Mode

Choose $\mathbf{Q} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$, then

$$\mathbf{H} = [1, \quad 0.4583, \quad 0.1]$$

thus

$$\mathbf{K}_e = [0, \quad 0.9, \quad 0.4583]$$

choose $\delta = 1.5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [1.5000 \quad 0.6874 \quad 0.1500]$$

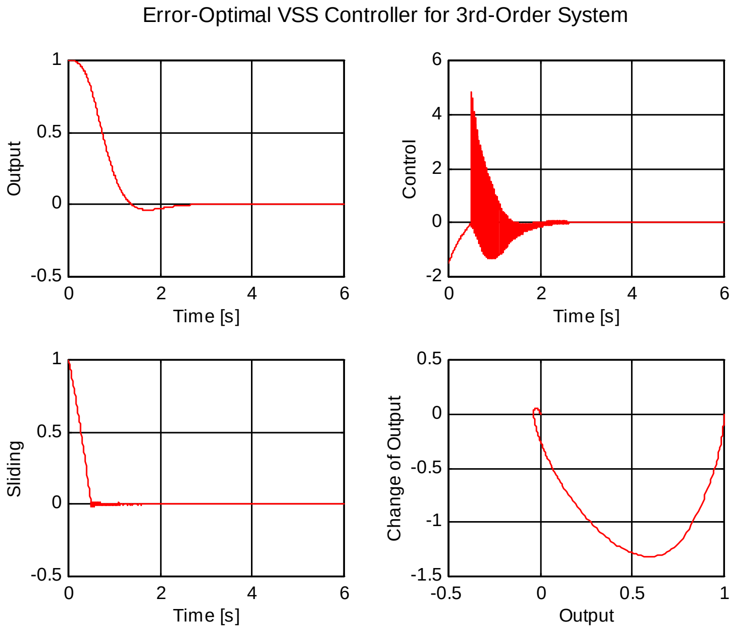

Fig. 3.30: Error-Optimal VSS Control for Example 3.4.

3.7.4.3. Performance compared with the Normal case

- I/O-State method: the same since the canonical system.
- Error-optimization: the normal approach is even better because it has no overshoot!

Remark 3.8: Comparison between Proposed Design and Nonlinear Hyperplane Design

In the original design, the author has proposed that the nonlinear hyperplane make the system responses faster. We still use our normal design and still get the same result, perhaps ours might be better because it is as fast as the original but with lower control effort!

3.7.5. Example 3.5: Efficiency of New Design

Consider the following DC servo system

$$G(s) = \frac{600}{s(s+20)}$$

so its state-space model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -20 \end{bmatrix} . \mathbf{x} + \begin{bmatrix} 0 \\ 600 \end{bmatrix} . u$$

Choose $\lambda_H = -60$ then

$$\mathbf{H} = \begin{bmatrix} 0.1 & 0.0017 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 0 & 0.0667 \end{bmatrix}$$

for all designs below.

3.7.5.1. Conventional Design

By the SMC design rule, choose $\delta = 0.4$, hence Theorem 3.2 and Corollary 3.1 yield

$$\mathbf{K}_r = \begin{bmatrix} 0.4 & 0.4 \end{bmatrix}$$

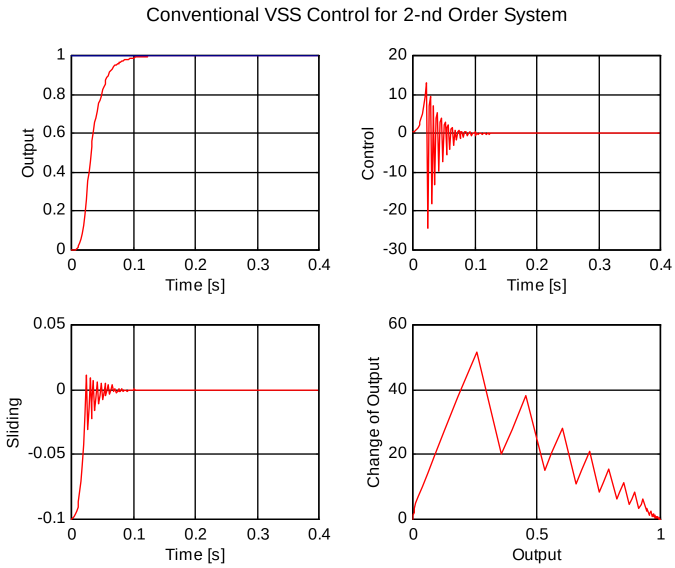

Fig. 3.31: Conventional VSS Control for Example 3.5 with $\delta = 0.4$

Choose $k_s = 50$, we have

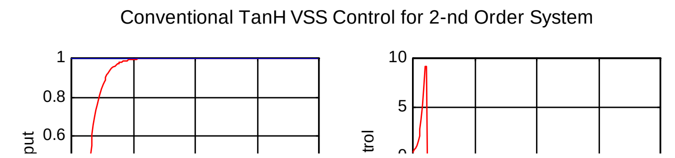

Fig. 3.32: Conventional TanH VSS Control for Example 3.5

3.7.5.2. New Design

By the SMC design rule, choose $\delta = 40$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [4.0000 \quad 0.0667]$$

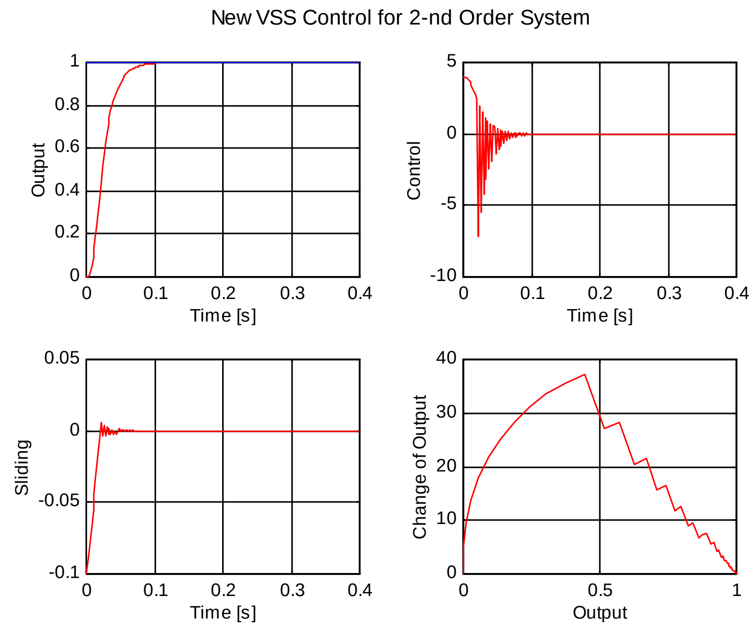

Fig. 3.33: New VSS Control for Example 3.5 with $\delta = 40$

Choose $k_s = 200$, we have

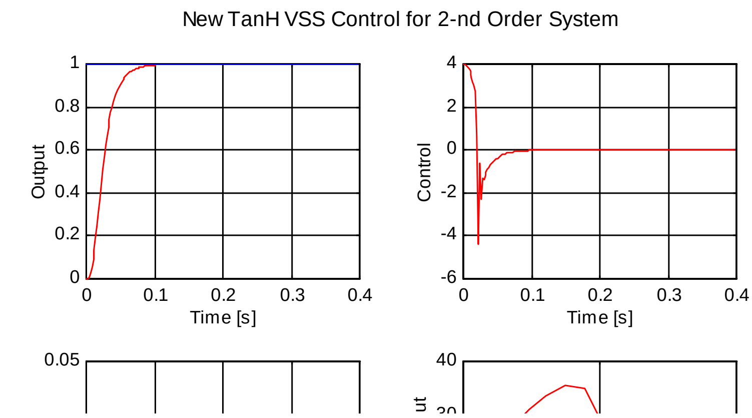

Fig. 3.34: New TanH VSS Control for Example 3.5

Remark 3.9: Performance of New Design

The new design is less chattering and lower control effort than the conventional approach. In addition, the new TanH VSS design has much higher sliding gain so less possibility of steady-state error. The reason is the system dynamics are very fast, these dynamics are include in $\mathbf{H}$ where $\mathbf{H}(1)/\mathbf{H}(2)=60$, the new design includes this fact in the reaching control where the conventional design ignores them and uses the same sliding margin for all system states.

3.7.6. Example 3.6: Error/Energy-Optimal VSS

Consider a system in Sivaramakrishnan *et al.* 1984

$$\dot{\mathbf{x}}=\begin{bmatrix}-0.05 & 6 & 0 & 0\\ 0 & -3.333 & 3.333 & 0\\ -5.208 & 0 & -12.5 & -12.5\\ 0.6 & 0 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\0\\12.5\\0\end{bmatrix}.u$$

choose

$$\mathbf{Q}=\begin{bmatrix}10 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 0.1 & 0\\ 0 & 0 & 0 & 0.1\end{bmatrix},\quad \text{and}\quad \mathbf{R}=1$$

then by the error/energy optimal sliding mode method, a hyperplane can determined as

$$\mathbf{H}=\left[0.04,\quad 0.0649\quad 0.08,\quad 0.0543\right]$$

thus

$$\mathbf{K}_e=\left[-0.3861,\quad 0.0259,\quad -0.7835,\quad -1\right]$$

choose $\delta=7$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r=\left[0.2828\quad 0.4543\quad 0.5600\quad 0.3801\right]$$

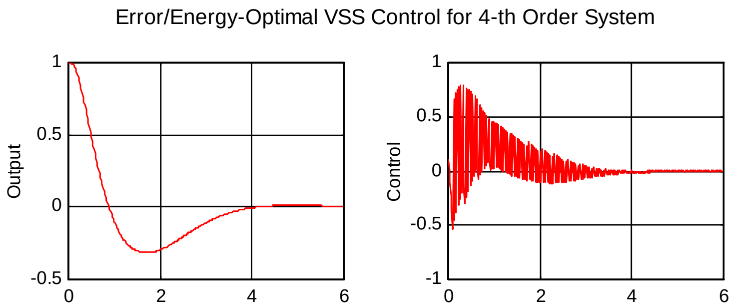

Fig. 3.35: Error/Energy-Optimal VSS Control for Example 3.6.

The control effort is very low as expected, however the system response is slow and there is a large overshoot.

3.7.7. Example 3.7: Applications of I/O-State Method (Section 2.3.4)

As mentioned in the previous chapter on the hyperplane design, the following examples are to illustrate the validity of an application of the I/O-State technique to design a hyperplane (Section 2.3.4). When the relative degree is less than the system order, the stability test in Section 3.2 must be used in the first place. Note that this method is applicable for both linear and nonlinear systems. The following illustrations are for linear systems to investigate into the I/O-state method, Chapter 6 will be for nonlinear systems.

3.7.7.1. Relative Degree less than System Order

By the stability criterion, we can have the following 2 cases

(a) Instability found by Stability Criterion: I/O-State Method Inapplicable

Consider a system in Spurgeon 1991

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -0.277 & -32.98 & -5.432 \\ 0.365 & -0.319 & -9.49 \\ 0 & 0 & -5 \end{bmatrix} . \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} . u$$

by the normal method, choose $\lambda_H = [-3, \quad -3]$, by the direct calculation technique, we have

$$\mathbf{H} = [-0.0018 \quad -0.0559 \quad 0.1]$$

thus

$$\mathbf{K}_e = [-0.0199 \quad 0.0783 \quad 0.0404]$$

choose $\delta = 2$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [0.0037 \quad 0.1118 \quad 0.2000]$$

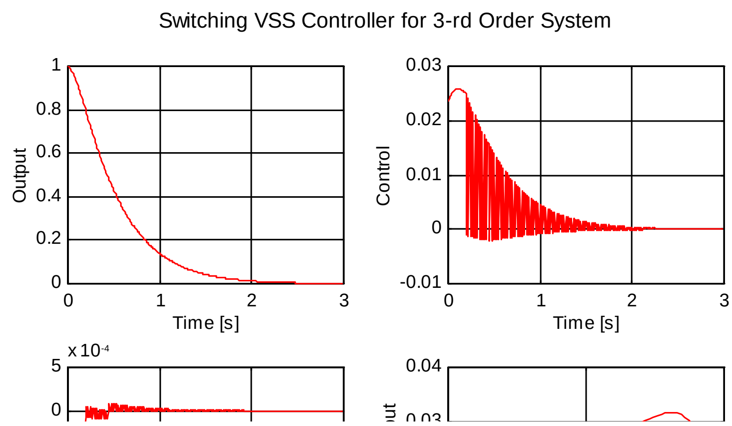

Fig. 3.36: Switching VSS Control for Example 3.7.

By the I/O-state method with a relative degree *r*, choose the same λ_H above, we have

$$\mathbf{H} = [-0.0501 \quad 0.6071 \quad 0.1000]$$

thus

$$\mathbf{K}_e = [0.2355 \quad 1.4596 \quad -5.9895]$$

so

$$n = 3, \quad r = 2 \quad \Rightarrow \quad \lambda_S = [-3, \quad 57.2989]: \textit{unstable!}$$

choose $\delta = 2$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [0.1003 \quad 1.2143 \quad 0.2000]$$

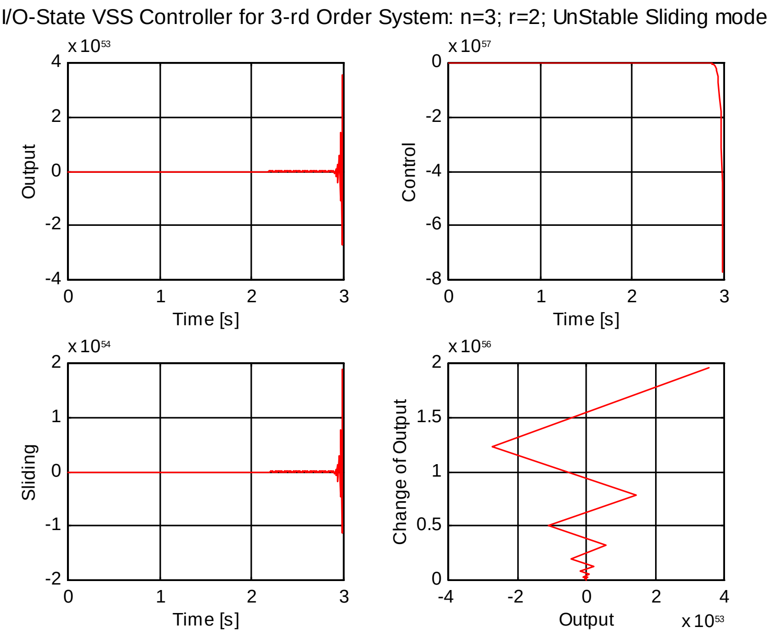

Fig. 3.37: I/O-State VSS Control for Example 3.7: *n*=3, *r*=2; Unstable Sliding Mode

This method fails in this case.

(b) Stability found by Stability Criterion: I/O-State Method Applicable

In Example 3.2, we had $n = 4$, $r = 3$, the relative degree is less than the system order, so the sliding eigenvalues need be checked

$$\lambda_S = [-6 \quad -6]$$

since they are stable, the I/O method is applicable.

3.7.7.2. Relative Degree equal to System Order

In Examples 3.1, 3.3 and 3.4, we had $n = r$, so the I/O method is always applicable as the system stability is guaranteed.

3.7.8. Example 3.8: A Simple Test of SMC Design Rule (Section 3.4)

Consider a system in White *et al.* 1984 as in Example 3.1.

3.7.8.1. Violation 1 in SMC Design Rule: Sliding Time much smaller than Reaching Time

The sliding margin is kept the same, $\delta = 2.5$, instead of $\lambda_H = [-7 \quad -7]$, the hyperplane eigenvalues are increased to violate the design rule

$$\lambda_H = [-35 \quad -35]$$

thus $$\mathbf{H} = \begin{bmatrix} 1225 & 70 & 1 \end{bmatrix}$$

and $$\mathbf{K}_e = \begin{bmatrix} -6 & 1214 & 64 \end{bmatrix}$$

with $\delta = 2.5$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = \begin{bmatrix} 3062.5 & 175 & 2.5 \end{bmatrix}$$

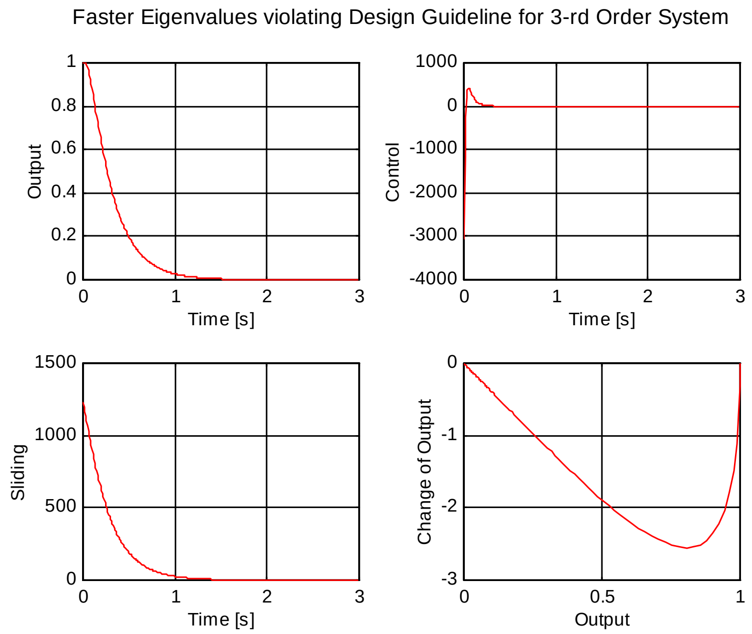

Fig. 3.38: Cf. Fig. 3.4, Switching VSS Control violating SMC Design Rule for Example 3.8

DISCUSSION: the eigenvalues are chosen about 5 times faster, the time response is the same (about 1.2 secs), but the maximum control effort is much higher (3000 compared to 220).

3.7.8.2. Violation 2 in SMC Design Rule: Reaching Time much smaller than Sliding Time

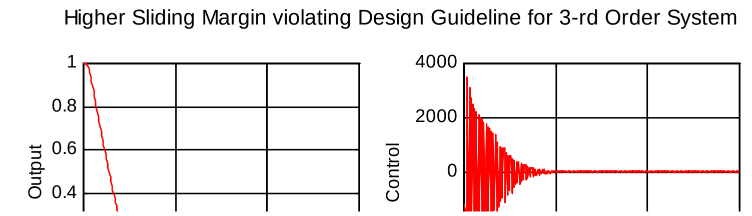

Fig. 3.39: Cf. Fig.3.4, Higher Sliding Margin violating SMC Design Rule for Example 3.7.

The hyperplane eigenvalues are kept the same, $\lambda_H = [-7 \quad -7]$, thus the hyperplane and the equivalent control are unchanged

$$\mathbf{H} = [49 \quad 14 \quad 1]$$

thus

$$\mathbf{K}_e = [-6 \quad 38 \quad 8]$$

instead of $\delta = 2.5$, the sliding margin is increased to violate the design rule, choose $\delta = 25$, hence Theorem 3.2 and Corollary 3.2 yield

$$\mathbf{K}_r = [1225 \quad 350 \quad 25]$$

DISCUSSION: the time response is a bit faster (1 sec compared to 1.2 secs), however the maximum control effort is much higher (3800 compared to 220)

3.7.9. Example 3.9: Steady-State Error when Reducing Sliding Gain

Consider a system in Lee *et al.* 1991 as in Example 3.3 where the sliding gain k_s of saturate, unitvector and tanh are reduced to unnecessarily low to see steady-state errors.

3.7.9.1. Saturate VSS

Choose $k_s = 1$, instead of $k_s = 50$

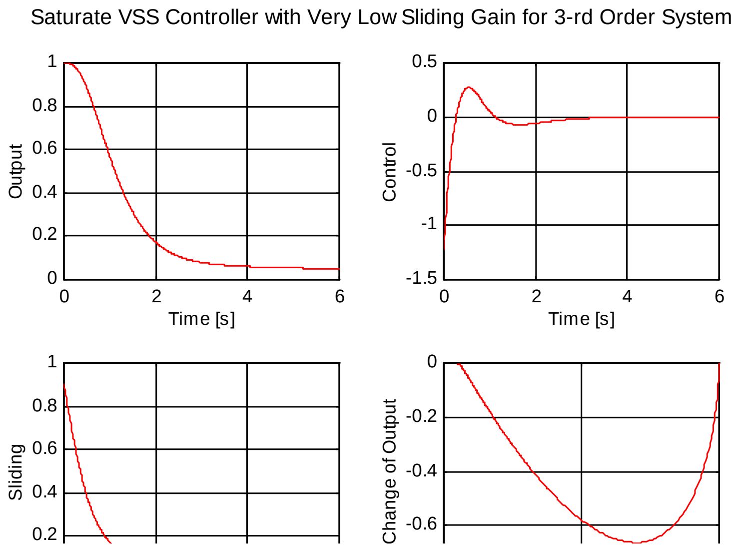

Fig. 3.40: Saturate VSS Control with Very Low Sliding Gain for Example 3.9.

3.7.9.2. Unitvector VSS

Choose $k_s = 1$, instead of $k_s = 100$

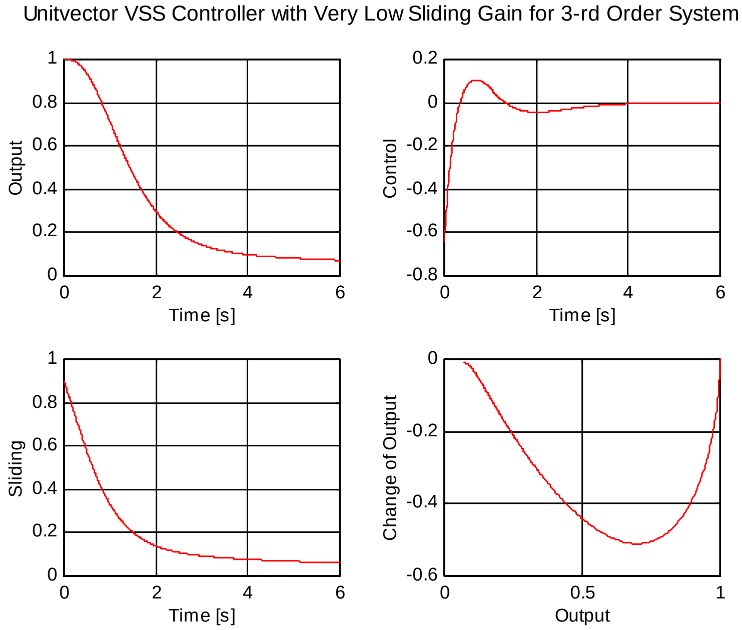

Fig. 3.41: Unitvector VSS Control with Very Low Sliding Gain for Example 3.9.

3.7.9.3. TanH VSS

Choose $k_s = 1$, instead of $k_s = 75$

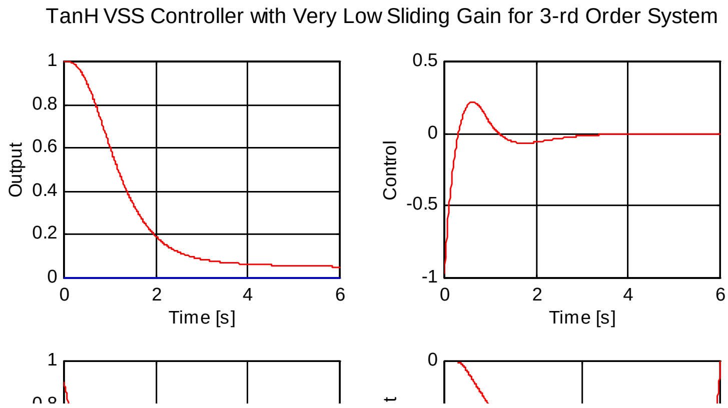

Fig. 3.42: TanH VSS Control with Very Low Sliding Gain for Example 3.9.

3.7.9.4. Discussion:

There are steady-state errors when the sliding gains are reduced much more than necessary.

3.7.10. Example 3.10: Integral VSS Control using Integral of Sliding Variable

In Example 3.3, the reference input is equal to 0, if a unit reference input is used then there is a noticeable steady-state error. We will use an integral VSS in Theorem 3.3 to eliminate this error.

3.7.10.1. Original Design

Consider a system in Hong *et al.* 1989 as in Example 3.3

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.\mathbf{x} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}.u$$

with the hyperplane

$$s = \mathbf{Hx}, \qquad \mathbf{H} = \begin{bmatrix} 0.7, & 1 \end{bmatrix}$$

and the original control function

$$u = k_1 x_1 - 0.43 x_2, \qquad k_1 = \begin{cases} 0.6, & \text{if } s x_1 > 0 \\ -1.26, & \text{if } s x_1 < 0 \end{cases}$$

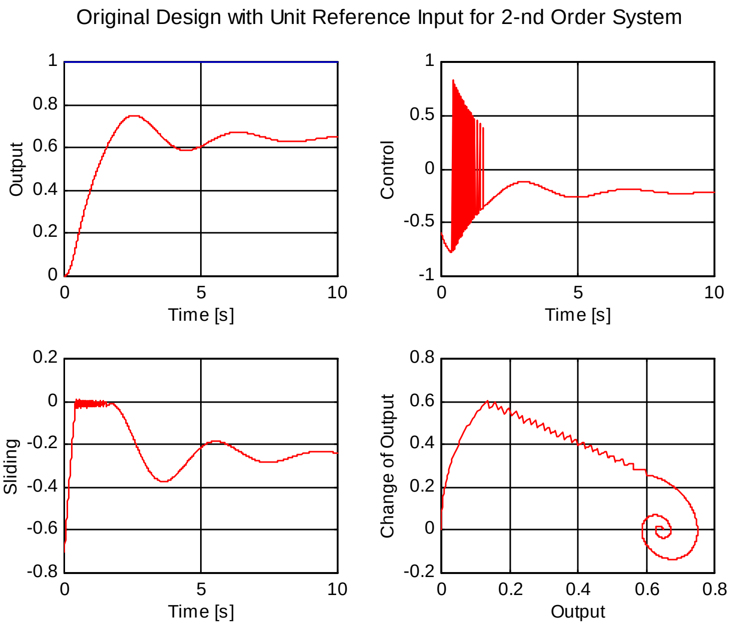

Fig. 3.43: Original Design with Unit Reference Input in Example 3.10: *Steady-State Error*

3.7.10.2. New Design

Choose the same hyperplane-eigenvalue as in Example 3.3, based on the SMC design rule, now the sliding margin is changed to $\delta = 1$ and to include an integral sliding margin of $\delta_i = 0.15$ in Theorem 3.3

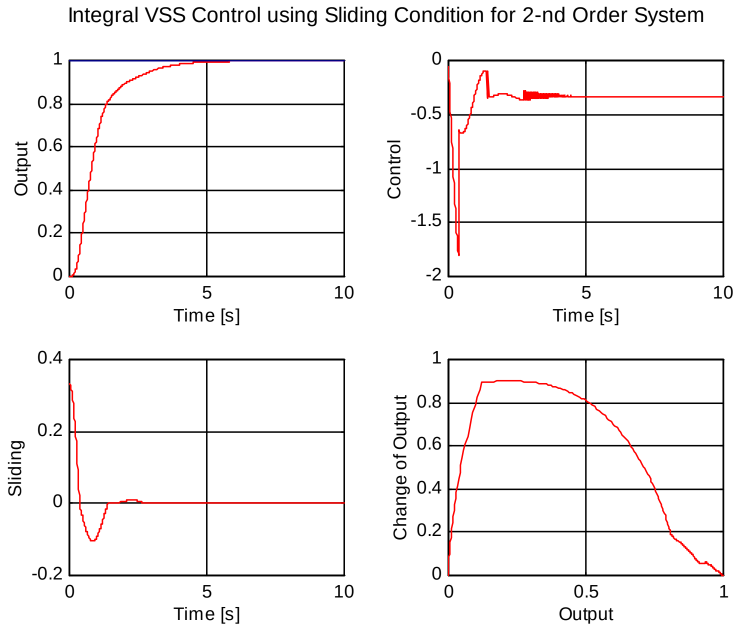

Fig. 3.44: Integral VSS Control using Integral Sliding Condition in Example 3.10: *No Steady-State Error*

Recall that the integral sliding condition in Eq.(3.11) has been used instead of the conventional sliding condition $s.\dot{s} < 0$.

3.7.10.3. Discussion

Steady-state error is eliminated.

3.7.11. Example 3.11: Integral VSS Control using Integral of Output Error

Consider the following system from Lin *et al.* 1992 where perturbation is ignored for the time-being and it will be considered in the next chapter

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -25 & -3 \\ -13 & 84 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -48 \\ -58 \end{bmatrix} u$$

3.7.11.1. Original Design

Using the linear quadratic optimal control, an original control has been

$$u = -1.078\, x_1 + 4.819\, x_2$$

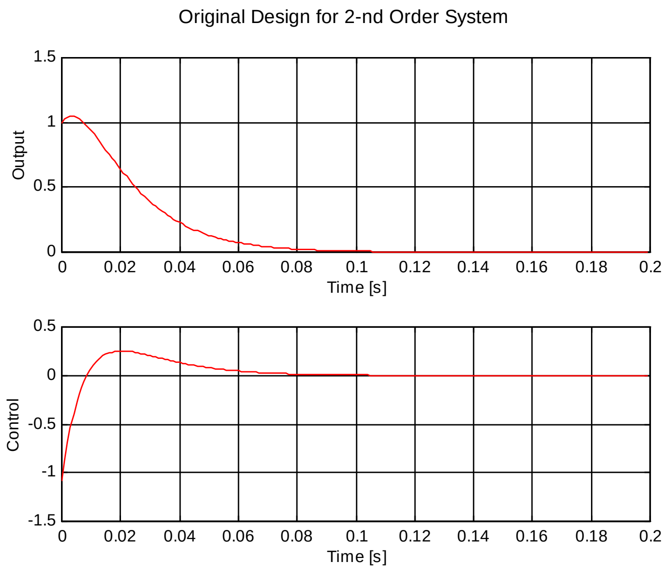

Fig. 3.45: Original Design in Example 3.11.

For a unit step response, we have

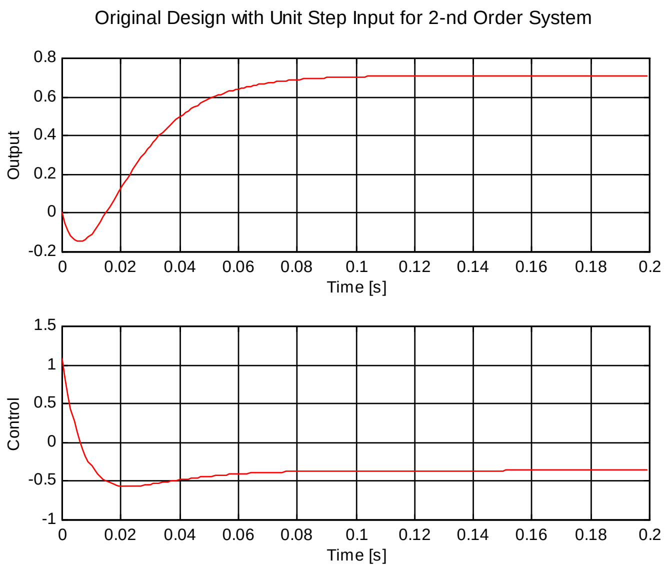

Fig. 3.46: Original Design with Unit Step Input in Example 3.11: *Steady-State Error.*

If the integral sliding condition is used, choose

$$\lambda_H = [-50] \quad \Rightarrow \quad \mathbf{H} = [0.0073 \quad -0.0233]$$

then Theorem 3.3 yields

$$u = -\delta_i \int_0^t s \, dt - \mathbf{K}_e \, \mathbf{x} - \delta \, \|\mathbf{x}\|_1 \, \mathrm{sgn}(s)$$

where

$$\mathbf{K}_e = [0.12, \quad -1.98], \qquad \delta = 50, \qquad \delta_i = 10^6$$

Note that δ_i in Theorem 3.3 is chosen as large as possible to reduce the steady-state error as much as possible.

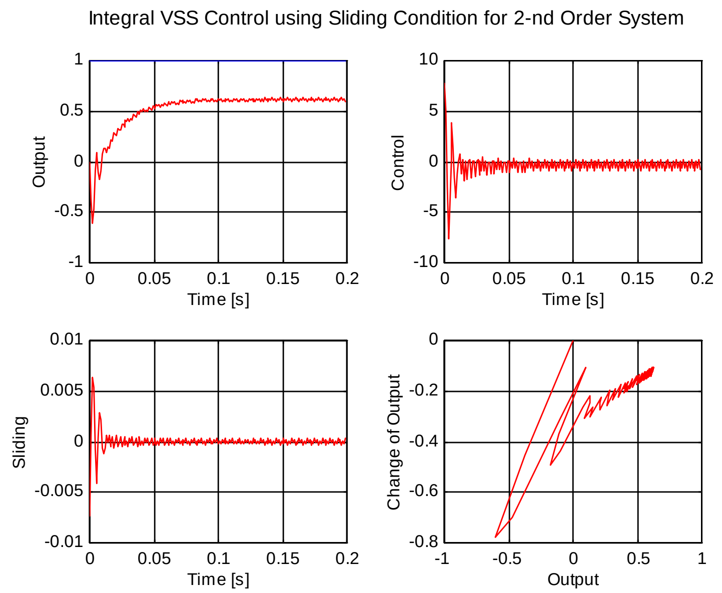

Fig. 3.47: Integral VSS Control using Integral Sliding Condition in Example 3.11: *Steady-State Error*

Note that the integral sliding condition in Eq.(3.11) has been used instead of the conventional sliding condition.

3.7.11.2. New Design

Choose the same hyperplane-eigenvalue as above, recall that the system order is augmented

$$\lambda_H = \left[-50, \quad -50\right] \quad \Rightarrow \quad \mathbf{H}_i = \left[0.5944, \quad 0.0234, \quad -0.0366\right]$$

for a unit step input, then Theorem 3.4 yields

$$u = 0.5944\, r - \mathbf{K}_e \mathbf{x}_i - \delta \left\|\mathbf{x}_i\right\| \operatorname{sgn}(s_i), \qquad \mathbf{x}_i = \left[\int_0^t (x_1 - r)\, dt, \quad x_1, \quad x_2\right]^T$$

where $$\mathbf{K}_e = \left[0, \quad 0.4856 \quad -3.1433\right], \qquad \delta = 0.1$$

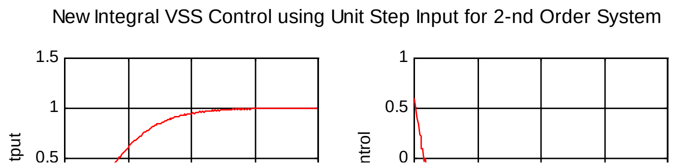

Fig. 3.48: New Integral VSS Control with Unit Step Input for Example 3.11: *No Steady-State Error*

Note that the sliding variable of an augmented system in Eq.(3.14) has been used instead of the sliding variable of the original system. The magnitude of s seems to increase and the sliding condition is thus not satisfied, but it is not the case due to the chattering and its maximum magnitude is less than 5×10^{-4} .

3.7.11.3. Discussion

The control effort in new integral VSS control is comparable to the case of regulator in Fig.3.47, but there is no steady-state error, cf. Fig.3.48.

3.8. CONCLUSION

In the VSS literature so far, the designs have been obtained in different techniques, all the VSS control functions have to satisfy the sliding condition $s.\dot{s} < 0$. The range is too large to choose a negative number for the inequality above. For SISO *discontinuous VSS* designs are White *et al.* 1984 where a control function is designed to satisfy the sliding condition with a given hyperplane (Example 3.1), Sivaramakrishnan *et al.* 1984 to apply the hyperplane design by partition of Utkin *et al.* 1978 (Example 3.2), Hong *et al.* 1989 with a proposed equal excursion sliding mode design to get an ideal sliding mode (Example 3.3), Lee *et al.* 1991 to propose a nonlinear hyperplane for a fast system response (Example 3.4). For *continuous pseudo-VSS* designs are Slotine *et al.* 1983 where a saturate function is proposed, Ambrosino *et al.* 1984 and Spurgeon 1991 with a unitvector function.

In this chapter, a unified VSS controller design has been developed. It is in a simple and unified manner with a SMC design rule applicable to all systems (*linear* or *nonlinear, SISO* or *MIMO*) and all approaches (*continuous* or *discontinuous VSS* or *continuous pseudo-VSS, continuous-time* or *discrete-time VSS, conventional* or *integral VSS,* and *sliding-mode fuzzy control*) . We have proposed some new features below

- Different from the current stability criterion, we have analyzed this stability problem based on the mechanism of the SMC, then we have proposed a simpler stability criterion for practical test applicable to SMC.
- We propose a SMC design rule to choose a sliding margin to guarantee the existence of the sliding mode for the invariance property. This rule is very effective both in simulations and in experiments (Chapter 8). There is a simple illustration in Example 3.7.

- The standard *if*-form in Utkin 1977 is a convenient form to write a control function that satisfies the sliding condition. Our derivation of a compact form is simple, valid and unified for linear or nonlinear systems with or without perturbation (to be used in Chapter 4, 7). First, based on the sliding condition we obtain the standard *if*-form, then by Lemma 3.1 we achieve the compact form;

- The new design has been proposed to include the system dynamics into the reaching control. The conventional design may fail in fast systems since it uses the same sliding margin for all system states regardless the system dynamics. The faster are the system dynamics, the better are the new designs in terms of lower control effort and higher sliding gain, that is less potential steady-state error (Example 3.1 to 3.5)

- There are 2 components in a conventional VSS control, they are reaching and sliding controls. The proposed integral VSS control is composed of 3 components where the first two are exactly the same as those in a conventional VSS control and the third is an additional integral control. The proposed integral VSS control with augmented system order has been shown to be efficient. It can cope with limitation using the integral sliding condition in Chang 1991 (Example 3.11) and solve the problem of pseudo-SMC using integral of error in Chern *et al.* 1991.

- The proposed analysis of the chattering problem is more comprehensive and precise than the works in Slotine *et al.* 1983, Ambrosino *et al.* 1984, Spurgeon 1991. Moreover our approach is in a unified manner for the saturate function and unitvector function. In addition, the introduction of the hyperbolic tangent to replace the saturate and unitvector functions for more convenient since TanH is a pre-defined mathematical function. The performances of the sliding functions are also analyzed and verified by simulations in Example 3.9 and by experimental results in Chapter 8. Note that this approach is a pseudo-VSS (Remark 3.4).

- For the I/O-state method, we have seen that it is crucial to do a stability test (Section 3.2) when the relative degree is less than the system order.

- In Section of numerical examples, all original designs in different approaches have been represented to compare with our proposed design in a unified approach. The following is a table to summarize the results on the performance in all the numerical examples above. The I/O-state method and the optimal method can be implemented in all methods (switching, saturate, unitvector and TanH VSS). For simplicity, the I/O-state method and the optimal method are implemented in the switching VSS control, so they may be compared to the equivalent switching control the normal case

	I/O-state	**Optimal I**
Ex 3.1	same as Switching	same as Switching
Ex 3.2	better than Switching	better than Switching
Ex 3.3	same as Switching	same as Switching
Ex 3.4	same as Switching	same as Switching

Chapter 3

Variable-Structure System Controller Design

Chapter 4

A New Robust Sliding Mode Controller-Observer Design

4.1. INTRODUCTION

In the current sliding mode control (SMC) literature, robust discontinuous SMC's have been developed for matched uncertain dynamical decouplable systems in Iyer *et al.* 1988 and for matched uncertain dynamical canonical systems in Emali *et al.* 1992. In Zhou *et al.* 1992 and Spurgeon *et al.* 1993, robust continuous pseudo-SMC with nonlinear control functions were developed for matched and unmatched uncertain systems. The former used a high order power functions and the latter used a unitvector function in place of a sign function. For these controllers, the sliding condition was satisfied with a boundary layer.

In this chapter, a new robust SMC design is fully presented in a unified manner for both discontinuous SMC (VSS) and continuous SMC control functions under both matched and unmatched uncertainties. The VSS design proposed in Chapter 3 is extended to deal with uncertain systems for VSS control functions. The control function is partitioned into 3 components: equivalent control, reaching control, and perturbation control. In the proposed linear SMC design, the control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem in the standard VSS design. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework. For example, the sliding margin can be shown to relate to the system eigenvalues.

As a SMC is based on a state-space model, an I-action may be required to eliminate a steady-state error. We will prove that the integral VSS control (discontinuous SMC) in Theorem 3.4 & 3.5 can be still applied to both robust discontinuous and linear SMC controls in this chapter.

Since the principal operating mode of a VSS control is the sliding mode, the VSS control can be seen as a subset of the SMC using discontinuous control functions.

Similar to state-space control design, SMC usually requires an observer for estimation of system states. In Bondarev *et al.* 1985, a linear Luenberger observer has been used as an observer of a VSS controller for deterministic linear systems (no uncertainties). In Walcott *et al.* 1987 and Yaz *et al.* 1993, a Lyapunov sliding condition has been used to design an observer for a class of systems under matched uncertainty restricted to a certain system structure. In Slotine *et al.* 1987, a sliding patch condition has been used to have a region of direct attraction where uncertainty is not fully tackled. In fact, in Walcott *et al.* 1987, Slotine *et al.* 1987 and Yaz *et al.* 1993, to cope with uncertainty, a Lyapunov sliding condition has been employed to include a switching component into a linear Luenberger observer where a linearized model is used for a nonlinear system. In Edwards *et al.* 1995, an observer has been implemented using output feedback technique.

In this chapter, we propose a novel robust sliding mode observer design for a wide class of systems under both matched and unmatched uncertainty. This design is a dual extension of the new robust sliding mode controller design mentioned above.

Based on the work on invariance property in Drazenovic 1969, we prove that the matching condition is necessary and sufficient condition. They are *valid for both discontinuous and continuous SMC.*

4.2. ROBUSTNESS IN SMC

In real dynamical systems, it is impossible to avoid uncertainties (due to imperfect modelling, due to the environment such as temperature, pressure...) and external disturbances. So the crucial demand is a solution to the *robust control* problem for uncertain dynamical systems. In the SMC literature, it is generally known that the main feature of SMC is its invariance to perturbations. This section shows what this "invariance" really means.

4.2.1. Invariance Condition to Uncertainties

Theorem 4.1: Invariance Condition to Uncertainty

Consider an uncertain dynamical system

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}.\mathbf{x} + \mathbf{B}.u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \mathbf{B}.u \tag{4.1}$$

where

$$\mathbf{x}, \mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\tilde{\mathbf{A}} \in \Re^{n\times n}, \quad u \in \Re$$

the *sliding mode is invariant to parameter uncertainties* if and only if the variation $\Delta\tilde{\mathbf{A}}$ satisfies the matching condition

$$\underline{\underline{\Delta\tilde{\mathbf{A}} = \mathbf{B}.\mathbf{z},}} \quad \forall \mathbf{z} \in \Re^{1\times n} \tag{4.2}$$

Proof

Consider a system

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \mathbf{B}u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \mathbf{B}u$$

and a hyperplane

$$s = \mathbf{H}\mathbf{x}$$

so, from Theorem 2.2, the sliding dynamics can be determined by

$$s = \mathbf{H}\mathbf{x} = 0 \quad \Rightarrow \quad \dot{\mathbf{x}} = \left[\mathbf{I} - \mathbf{B}(\mathbf{H}\mathbf{B})^{-1}\mathbf{H}\right].\tilde{\mathbf{A}}\mathbf{x}$$

or

$$\dot{\mathbf{x}} = \left[\mathbf{I} - \mathbf{B}(\mathbf{H}\mathbf{B})^{-1}\mathbf{H}\right].\left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} \tag{4.3}$$

(a) Necessary Condition (If Clause)

By Eq.(4.3), the sliding dynamics are *invariant* if

$$\left[\mathbf{I} - \mathbf{B}(\mathbf{H}\mathbf{B})^{-1}\mathbf{H}\right].\Delta\tilde{\mathbf{A}}.\mathbf{x} = 0$$

for any **x**, so we obtain 2 cases as follows

$$\left[\mathbf{I}-\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\right].\Delta\tilde{\mathbf{A}}.\mathbf{x}=0 \Rightarrow \begin{cases}\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}=\mathbf{I} & : \quad (\alpha)\\ \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x}=\Delta\tilde{\mathbf{A}}.\mathbf{x}: & (\beta)\end{cases}$$

- Case $(\alpha): \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}=\mathbf{I}$

$$\mathbf{B}=\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix},\qquad \mathbf{H}=\begin{bmatrix}h_1 & h_2 & \cdots & h_n\end{bmatrix},\qquad (\mathbf{HB})^{-1}=\frac{1}{b_1h_1+b_2h_2+\cdots+b_nh_n}=c\neq 0$$

thus

$$\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}=c\mathbf{BH}=c.\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}.\begin{bmatrix}h_1 & h_2 & \cdots & h_n\end{bmatrix}=c.\begin{bmatrix} b_1h_1 & b_1h_2 & \cdots & b_1h_n\\ b_2h_1 & b_2h_2 & \cdots & b_2h_n\\ \vdots & \vdots & \ddots & \vdots\\ b_nh_1 & b_nh_2 & \cdots & b_nh_n\end{bmatrix}$$

hence

$$c\begin{bmatrix} b_1h_1 & b_1h_2 & \cdots & b_1h_n\\ b_2h_1 & b_2h_2 & \cdots & b_2h_n\\ \vdots & \vdots & \ddots & \vdots\\ b_nh_1 & b_nh_2 & \cdots & b_nh_n\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix};$$

by $c\neq 0$,then

$$\left.\begin{array}{l} cb_ih_i=1\Rightarrow b_i\neq 0 \quad \text{AND} \quad h_i\neq 0, \quad \forall\, i\in[1,n]\\ cb_ih_j \underset{i\neq j}{=} 0\Rightarrow b_i=0 \quad \text{OR} \quad h_j=0, \quad \forall\, i,\forall\, j\in[1,n]\end{array}\right\} : \textbf{contradiction!}$$

- Case (β) : $\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x}=\Delta\tilde{\mathbf{A}}.\mathbf{x}$, then there exists at least one solution

$$\Delta\tilde{\mathbf{A}}=\mathbf{B}.\mathbf{z},\qquad \mathbf{z}\in\Re^{1\times n}$$

(b) Sufficient Condition (Only-If Clause)

Eq.(4.2) gives

$$\left[\mathbf{I}-\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\right].\Delta\tilde{\mathbf{A}}=\left[\mathbf{I}-\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\right].\mathbf{Bz}=[\mathbf{B}-\mathbf{B}].z=\mathbf{0}$$

thus the sliding mode is invariant by Eq.(4.3).

Q.E.D.

Remark 4.1: Form of Matching Condition

Note that **z** takes an arbitrary value, so the matching condition requires the form rather the actual value.

Corollary 4.1: Alternative Form of Matching Condition

The system

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u$$

is under matching condition if

$$\underline{\underline{\Delta\tilde{\mathbf{A}}.\,\mathbf{x} = \mathbf{B}.\,v_1,}} \qquad \forall v_1 \in \Re \tag{4.4.a}$$

and

$$\underline{\underline{\Delta\tilde{\mathbf{B}} = v_2\,.\,\mathbf{B},}} \qquad \forall v_2 \in \Re \tag{4.4.b}$$

then

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(u + v) \tag{4.4.c}$$

Proof:

Post-multiply **x** to the matching condition in Eq.(4.2), we have

$$\Delta\tilde{\mathbf{A}}.\,\mathbf{x} = \tilde{\mathbf{B}}.\,\mathbf{z}.\,\mathbf{x} = \tilde{\mathbf{B}}.\,v_1$$

since **x** and **z** are arbitrary, so is v_1. We also have

$$\Delta\tilde{\mathbf{B}}.\,u = \mathbf{B}.\,v_2\,.\,u = \mathbf{B}.\,v_3$$

thus Eq.(4.4.c) is achieved.

Q.E.D.

Corollary 4.2: Canonical System with Matching Condition

Any canonical system satisfies the matching uncertainty condition.

Proof

Consider an uncertain dynamical canonical system

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n \end{bmatrix}, \quad \Delta\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \Delta a_1 & \Delta a_2 & \Delta a_3 & \cdots & \Delta a_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{b}_n \end{bmatrix}, \quad \Delta\tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta b_n \end{bmatrix}$$

thus the matching condition is satisfied, since

$$\Delta\tilde{\mathbf{A}}.\,\mathbf{x} + \Delta\tilde{\mathbf{B}}.\,u = \mathbf{B}.\,v$$

Q.E.D.

4.2.2. Invariance Condition to Disturbances.

Theorem 4.2: Sufficient Invariance Condition to Disturbance

Consider a linear system under disturbances

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u + \mathbf{W}.v$$

the sliding mode is *invariant to disturbances* if and only if the matching condition is satisfied

$$\underline{\underline{\mathbf{W} = \mathbf{B}.z}}, \quad \forall z \in \Re \tag{4.5}$$

where

$$\mathbf{x}, \mathbf{B}, \mathbf{W} \in \Re^{n\times 1}, \quad \mathbf{A} \in \Re^{n\times n}, \quad u, v \in \Re$$

Proof

In the sliding mode

$$\mathbf{Hx} = 0 \Rightarrow \mathbf{H}\dot{\mathbf{x}} = 0 \Rightarrow \mathbf{H}(\mathbf{Ax} + \mathbf{B}.u + \mathbf{W}.v) = 0 \Rightarrow \mathbf{HAx} + \mathbf{HB}.u + \mathbf{HW}.v = 0$$

so the equivalent control

$$u_{eq} = -(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{Ax} + \mathbf{W}.v)$$

then the sliding equation

$$\begin{cases} \dot{\mathbf{x}} = \left[\mathbf{I} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\right].(\mathbf{Ax} + \mathbf{W}.v) \\ \mathbf{Hx} = 0 \end{cases}$$

thus, similarly to the case of uncertainties above, the sliding mode is invariant to disturbances if and only if

$$\left[\mathbf{I} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\right].\mathbf{W}.v = \mathbf{0} \Leftrightarrow \mathbf{W} = \mathbf{B}.z, \quad \forall z \in \Re$$

Q.E.D.

Remark 4.2: Invariance Property of Sliding Eigenvalues under Matching Condition

By Theorem 3.1 on a stability test, we have

- In case of the system without any perturbation, the sliding-eigenvalues are equal to the hyperplane-eigenvalues, hence the Hurwitz hyperplane-eigenvalues do guarantee the stability of the system.
- Under perturbations satisfying the matching condition, the sliding-eigenvalues are absolutely unchanged and equal to the hyperplane-eigenvalues, so the Hurwitz hyperplane-eigenvalues also do guarantee the stability of the system.

4.3. A New Robust VSS Controller Design (Discontinuous SMC)

A discontinuous control is also called a *switching control.* As mentioned in Chapter 3, there are 2 VSS control types, but only the first type can be extended into a robust VSS control. A robust VSS control function consists of 3 components, namely equivalent, reaching and perturbation controls, where the first two are similar to the normal VSS and the third additional one is to deal with perturbations. Different from the VSS literature so far (White *et al.* 1984, Sivaramakrishnan *et al.* 1984, Hong *et al.* 1989, Lee *et al.* 1991), our proposed design is extended from Chapter 3 where all designs are done in a unified manner.

First we propose an assumption which is practically satisfied in the real world

Assumption 4.1: System Constraint on Parametric Variation

A system matrix **B** takes any variation such that the polarity of $\left(\mathbf{H}\tilde{\mathbf{B}}\right)$ is unchanged, i.e.

$$\operatorname{sgn}\left(\mathbf{H}\tilde{\mathbf{B}}\right) = \operatorname{sgn}(\mathbf{HB}) \quad \Rightarrow \quad \left|\Delta\tilde{\mathbf{B}}\right| < \Delta\mathbf{B} \tag{4.6}$$

where

$$\tilde{\mathbf{B}} = \mathbf{B} + \Delta\tilde{\mathbf{B}}$$

Under Assumption 4.1, we propose the following theorem to design a robust VSS controller

Theorem 4.3: Robust VSS Controller Design under Uncertainty and Disturbance

For a linear system under perturbations (uncertainties and disturbances)

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}.\mathbf{x} + \tilde{\mathbf{B}}.u + \mathbf{W}.v = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u + \mathbf{W}.v$$

and a hyperplane

$$s = \mathbf{H}.\mathbf{x}$$

with

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| = \Delta\mathbf{B}, \qquad |v| = \bar{v}$$

then, under Assumption 4.1, there exists a constant $\delta > 0$ for a *VSS control function* to be determined as

$$\underline{\underline{u = u_e + u_r + u_p}} \tag{4.7}$$

where

- equivalent control

$$u_e = -\mathbf{K}_e.\mathbf{x}, \qquad \mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.7.a}$$

- reaching control

$$u_r = -(\mathbf{HB})^{-1}\mathbf{K}_r.|\mathbf{x}|.\operatorname{sgn}(s), \qquad \mathbf{K}_r = \delta.|\mathbf{H}| \tag{4.7.b}$$

- perturbations control

$$u_p = -\left(K_{p0} + \mathbf{K}_p.|\mathbf{x}|\right).\operatorname{sgn}(s), \qquad K_{p0} = \frac{|\mathbf{H}.\mathbf{W}|\bar{v}}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\operatorname{sgn}(\mathbf{HB})}, \qquad \mathbf{K}_p = \frac{|\mathbf{H}|.\Delta\mathbf{A}}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\operatorname{sgn}(\mathbf{HB})}, \qquad |\mathbf{x}| = \left[|x_i|\right] \tag{4.7.c}$$

with

$$\mathbf{x}, \mathbf{B}, \Delta\mathbf{B}, \mathbf{W} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s, v \in \Re, \quad \delta \in \Re_{(+)}$$

Proof

We have

$$\dot{s} = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}\tilde{\mathbf{A}}\mathbf{x} + \mathbf{H}\tilde{\mathbf{B}}u + \mathbf{HW}v$$

substituting Eq.(4.7) gives

$$\dot{s} = \mathbf{H}\tilde{\mathbf{A}}\mathbf{x} + \mathbf{HW}v - \frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\mathbf{HAx} - \frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\delta.|\mathbf{H}|.|\mathbf{x}|.\operatorname{sgn}(s) - \frac{\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\left(|\mathbf{H}.\mathbf{W}|\bar{v} + |\mathbf{H}|.\Delta\mathbf{A}.|\mathbf{x}|\right).\operatorname{sgn}(s)$$

$$\dot{s} = -\frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\delta.|\mathbf{H}|.|\mathbf{x}|.\operatorname{sgn}(s) + \left(\mathbf{HW}v - \frac{\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}|\mathbf{H}.\mathbf{W}|\bar{v}.\operatorname{sgn}(s)\right) + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} - \frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\mathbf{HAx} - \frac{\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}.|\mathbf{H}|.\Delta\mathbf{A}.|\mathbf{x}|.\operatorname{sgn}(s)$$

or

$$s\dot{s} = -|s|.\frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\delta.|\mathbf{H}|.|\mathbf{x}| - s\frac{\mathbf{H}\tilde{\mathbf{B}}}{\mathbf{HB}}\mathbf{HAx} + \left(s\mathbf{HW}v - |s|.\frac{|\mathbf{H}\tilde{\mathbf{B}}|}{\inf|\mathbf{H}\tilde{\mathbf{B}}|}|\mathbf{H}.\mathbf{W}|\bar{v}\right) + \left(s\mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} - |s|.\frac{|\mathbf{H}\tilde{\mathbf{B}}|}{\inf|\mathbf{H}\tilde{\mathbf{B}}|}.|\mathbf{H}|.\Delta\mathbf{A}.|\mathbf{x}|\right)$$

then the Assumption 4.1 guarantees the existence of a sliding margin $\delta > 0$ such that $s\dot{s} < 0$ for a bounded-input bounded-output system.

Q.E.D.

Remark 4.3: Robust VSS Controller Design under No Perturbations

If there is no perturbation, we have

$$\tilde{\mathbf{A}} = \mathbf{A}, \quad \tilde{\mathbf{B}} = \mathbf{B}, \quad \Delta\mathbf{A} = \mathbf{0}, \quad \Delta\mathbf{B} = \mathbf{0}, \quad \bar{v} = 0$$

then by Eq.(4.7.c), the perturbation control vanishes

$$u_p = 0$$

and the control function in Eq.(4.7) becomes

$$u = u_e + u_r$$

it is exactly the same as in Eq.(3.4) for the normal case without perturbation.

Remark 4.4: Equivalent Control in Robust VSS Controller

Consider a system under no perturbations

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u$$

and a hyperplane

$$s = \mathbf{Hx}$$

then, by Eq.(3.4), a discontinuous VSS control function can be determined by

$$u = u_e + u_r$$

with

$$u_e = (\mathbf{HB})^{-1}\mathbf{HAx} \qquad : \text{equivalent control}$$

$$u_r = (\mathbf{HB})^{-1}\delta.|\mathbf{H}|.|\mathbf{x}|.\mathrm{sgn}(s) : \text{reaching control}$$

Once in the sliding mode, $s = 0 \Rightarrow \mathrm{sgn}(s) = 0$, by definition, so

$$u = u_e$$

Now, under perturbation, **A** and **B** are unknown, so the equivalent control above, $u_e = \left(\mathbf{H}\tilde{\mathbf{B}}\right)^{-1}\mathbf{H}\tilde{\mathbf{A}}\mathbf{x}$, is undefined. It is impossible to design a control function by the above formula. By Eq.(4.7), it is possible to determine a control function since $u_e = -(\mathbf{HB})^{-1}\mathbf{HA}.\mathbf{x}$ is defined. This component is termed "equivalent control" since the control function reduces to this component once in the sliding mode, by definition it must be the equivalent control.

Corollary 4.3: Robust VSS Controller under Reduced Parametric Variation

If

$$\Delta\mathbf{B} = \mathbf{0}$$

then the *VSS control function* in Theorem 4.3 becomes

$$\underline{\underline{u = u_e + u_r + u_p}} \tag{4.8}$$

where

- equivalent control

$$u_e = -\mathbf{K}_e.\mathbf{x}, \quad \mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.8.a}$$

- reaching control

$$u_r = -(\mathbf{HB})^{-1}\mathbf{K}_r.|\mathbf{x}|.\operatorname{sgn}(s), \quad \mathbf{K}_r = \delta.|\mathbf{H}| \tag{4.8.b}$$

- perturbations control

$$u_p = -\left(K_{p0} + \mathbf{K}_p.|\mathbf{x}|\right).\operatorname{sgn}(s), \quad K_{p0} = (\mathbf{HB})^{-1}|\mathbf{H}.\mathbf{W}|\bar{v}, \quad \mathbf{K}_p = (\mathbf{HB})^{-1}|\mathbf{H}|.\Delta\mathbf{A}, \quad |\mathbf{x}| = \left[|x_i|\right] \tag{4.8.c}$$

with

$$\mathbf{x},\mathbf{B},\mathbf{W} \in \Re^{n\times 1}, \quad \mathbf{A},\Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u,s,v \in \Re, \quad \delta \in \Re_{(+)}$$

4.4. A NEW ROBUST LINEAR SMC DESIGN

First we start with a basic linear SMC without any perturbation where its control function is composed of 2 components: an equivalent control and a reaching control. Next we present a robust linear SMC under an un-matched perturbation where its control function is composed of 3 components: an equivalent control, a reaching control and a perturbation control. Then the matched perturbation is a special case of the un-matched one. To the best of our knowledge, there was only one work on the robust continuous SMC in Zhou *et al.* 1992.

Current Design	Proposed Design
• pseudo-SMC: derivation unavailable	• SMC: precise derivation
• non-linear continuous control function	• linear continuous control function
• both *sliding margin,* and *boundary layer* are used in design	• only *sliding margin* is used in design
• N. A.	• sliding margin and hyperplane-eigenvalues are system-eigenvalues
• N. A.	• design separation: equivalent control, reaching control, and perturbation control

4.4.1. Linear SMC

Different from the SMC literature so far (DeCarlo *et al.* 1988, Zhou *et al.* 1992), it is shown that a sliding margin and hyperplane-eigenvalues are the closed-loop system-eigenvalues. This is a criterion which is used to choose a sliding margin in designing a linear SMC control function.

Theorem 4.4: Linear SMC Design

For a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

with a hyperplane

$$s = \mathbf{H}\mathbf{x}$$

then a linear SMC function can be determined by

$$\underline{\underline{u = -\mathbf{K}\mathbf{x} = -(\mathbf{K}_e + \mathbf{K}_r).\mathbf{x}}} \tag{4.9}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.9.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\mathbf{H}.\delta \tag{4.9.b}$$

with

$$\mathbf{x},\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{A} \in \Re^{n\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re, \quad \delta \in \Re_{(+)}$$

Proof

From a hyperplane equation

$$\dot{s} = \mathbf{H}.\dot{\mathbf{x}} = \mathbf{H}(\mathbf{A}.\mathbf{x} + \mathbf{B}.u) = \mathbf{HB}.\left[(\mathbf{HB})^{-1}\mathbf{HAx} + u\right]$$

by the above control function

$$u = -\mathbf{Kx} = -(\mathbf{HB})^{-1}\mathbf{HAx} - (\mathbf{HB})^{-1}\underbrace{\mathbf{Hx}}_{s}.\delta;$$

hence

$$\dot{s} = -\delta.\mathbf{HB}.(\mathbf{HB})^{-1}.s = -\delta.s \Rightarrow s.\dot{s} = -\delta.s^2 \le 0$$

thus the sliding condition is satisfied. **Q.E.D.**

To design a SMC, we must choose δ. To do so, we propose the following theorem.

Theorem 4.5: Closed-Loop Eigenvalues composed of Hyperplane Eigenvalues and Sliding Margin

If a system has a sliding margin δ and a hyperplane-eigenvalue $\boldsymbol{\lambda}_H$

$$\boldsymbol{\lambda}_H \in \Re_{(-)}^{1\times(n-1)}$$

then the system closed-loop eigenvalues are determined by

$$\underline{\underline{\boldsymbol{\lambda}_C = \{-\delta, \quad \boldsymbol{\lambda}_H\} \in \Re^{1\times n}}} \tag{4.10}$$

Proof:

By definition of eigenvalues: $\mathrm{eig}(\mathbf{M}) = \left\{\lambda_i \middle| \det[\lambda_i\mathbf{I} - \mathbf{M}] = 0\right\}$, from (4.9.a)

$$\mathbf{A} - \mathbf{BK} = \mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}(\mathbf{HA} + \mathbf{H}.\delta) = \mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}.(\mathbf{A} + \delta.\mathbf{I})$$

To prove $-\delta$ is one of the eigenvalues, we have to prove

$$\det\left[(-\delta).\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right] = 0 \Rightarrow \det\left\{\left[(\mathbf{HB})^{-1}\mathbf{BH} - \mathbf{I}\right].(\mathbf{A} + \delta.\mathbf{I})\right\} = 0 \tag{4.11}$$

By the properties of a determinant

$$|\mathbf{P}.\mathbf{Q}| = |\mathbf{P}|.|\mathbf{Q}| = |\mathbf{Q}|.|\mathbf{P}|, \quad \mathbf{P},\mathbf{Q} \in \Re^{n\times n}$$

and

$$|\mathbf{I}_n + \mathbf{P}.\mathbf{Q}| = |\mathbf{I}_m + \mathbf{Q}.\mathbf{P}|, \quad \mathbf{P} \in \Re^{n\times m}, \quad \mathbf{Q} \in \Re^{m\times n}$$

then Eq.(4.11) is satisfied since

$$\left|(\mathbf{HB})^{-1}\mathbf{BH} - \mathbf{I}\right| = \left|(\mathbf{HB})^{-1}\mathbf{HB} - \mathbf{I}\right| = 0$$

Thus we can write

$$\left|\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA} + (\mathbf{HB})^{-1}\mathbf{H}.\delta\right| = (\lambda+\delta)\times\mathscr{P}^*\left(\lambda^{n-1}\right) \tag{4.12}$$

Eq.(4.12) holds for all δ , so it must hold for $\delta = 0$, then Eq.(4.12) becomes

$$\left|\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA}\right| = \lambda\times\mathscr{P}^*\left(\lambda^{n-1}\right) \tag{4.13}$$

From Eq.(2.10), we obtain

$$\left|\lambda\mathbf{I} - \left[\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA}\right]\right| = \lambda\times\mathscr{P}\left(\lambda^{n\text{-}1}\right) : \boldsymbol{\lambda}_H = \left\{\lambda_i \middle| \mathscr{P}\left(\lambda^{n\text{-}1}\right) = 0\right\} \tag{4.14}$$

by Eq.(4.13), we achieve

$$\mathscr{P}^*\left(\lambda^{n-1}\right) \equiv \mathscr{P}\left(\lambda^{n-1}\right)$$

this proves that $\boldsymbol{\lambda}_H$ are the remaining eigenvalues.

Q.E.D.

Remark 4.5: Reduced Order of Sliding Mode

In the sliding mode, the system states are sliding on the hyperplane. Eq.(4.10) shows that the sliding mode is reduced order.

4.4.2. Robust Linear SMC

In Section 4.3, both disturbances and uncertainties are simultaneously taken into account in designing a robust VSS control function (discontinuous SMC). To design a robust linear SMC function, due to mathematical derivations, disturbances and perturbations are considered separately.

For uncertain dynamical systems in Zhou *et al.* 1992 and Spurgeon *et al.* 1993, the control function is not only complicated, but is also pseudo-sliding mode since the sliding condition is not satisfied within the boundary layer (Remark 3.4). Still both the sliding margin and boundary layer are used in the design and it is not clear how to determine them. In fact, there is no proof for the proposed formulas! Moreover, the control function is nonlinear even for a linear system; and for both cases of disturbances and uncertainties, it is a *pseudo-SMC* only!

We propose an alternative approach to designing a linear SMC function for the case of uncertainties. It is simple for linear systems: only the sliding margin is used with the design rule (Section 3.4) and is not pseudo-SMC. Moreover, the control function is linear for a linear system and even for a certain class of non-linear systems. For the case of disturbances, we can get the pseudo-SMC only.

4.4.2.1. Robust Linear SMC under External Disturbances

Theorem 4.6: Robust Linear SMC Design under External Disturbance

Consider a linear system under external disturbances

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u + \mathbf{W}.v, \quad |v| \le \bar{v} \tag{4.15}$$

with a hyperplane equation

$$s = \mathbf{H}\mathbf{x}$$

where

$$\mathbf{x}, \mathbf{B}, \mathbf{W} \in \Re^{n\times 1}; \quad \mathbf{A} \in \Re^{n\times n}; \quad \mathbf{H} \in \Re^{1\times n}; \quad u, v, s \in \Re$$

then a robust linear SMC function can be found from

$$\underline{\underline{u = -\mathbf{K}.\mathbf{x}, \quad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p}} \tag{4.16}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.16.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\mathbf{H}.\delta \tag{4.16.b}$$

- perturbation control

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\delta_p\mathbf{H}.\left(\sup|\mathbf{HW}|.\bar{v}\right) \tag{4.16.c}$$

if the following disturbance condition is satisfied

$$|s| > \frac{\sup|\mathbf{HW}|.\bar{v}}{\delta + \delta_p\left(\sup|\mathbf{HW}|.\bar{v}\right)} \tag{4.17}$$

where δ_p is a sliding margin for disturbance.

Proof

From Eq.(4.16), we have

$$s.\dot{s} = s.\mathbf{HB}\left\{-(\mathbf{HB})^{-1}\mathbf{HAx} - (\mathbf{HB})^{-1}\left(\delta + \delta_p \sup|\mathbf{HW}v|\right).s + (\mathbf{HB})^{-1}\mathbf{HAx} + (\mathbf{HB})^{-1}\mathbf{HW}v\right\}$$

or

$$s.\dot{s} = s.\left\{-\left(\delta + \delta_p \sup|\mathbf{HW}v|\right).s + \mathbf{HW}v\right\} = -\left(\delta + \delta_p \sup|\mathbf{HW}v|\right).s^2 + \mathbf{HW}v.s$$

so

$$s.\dot{s} = s.\left\{-\left(\delta + \delta_p \sup|\mathbf{HW}v|\right).s + \mathbf{HW}v\right\} = -\left(\delta + \delta_p \sup|\mathbf{HW}v|\right).s^2 + \mathbf{HW}v.s$$

since

$$\sup|\mathbf{HW}v| = \sup|\mathbf{HW}|.\bar{v}$$

From

$$\left(\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}\right) > 0$$

we consider the following 2 cases

- if $\mathbf{HW}v > 0 \Rightarrow \dfrac{\mathbf{HW}v}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}} > 0$

s		0		$\dfrac{\mathbf{HW}v}{\delta + \delta_p \sup\lvert\mathbf{HW}\rvert.\bar{v}}$	
$\mathbf{s}.\dot{\mathbf{s}}$	–	0	+	0	–

- if $\mathbf{HW}v < 0 \Rightarrow \dfrac{\mathbf{HW}v}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}} < 0$

s		$\dfrac{\mathbf{HW}v}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}}$		0	
$\mathbf{s}.\dot{\mathbf{s}}$	–	0	+	0	–

Therefore, in the most conservative case, the sliding condition is satisfied when

$$|s| > \frac{|\mathbf{HW}|.\bar{v}}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}}$$

Q.E.D.

Remark 4.6: Disturbance Condition in Sliding Mode

The disturbance condition in Eq.(4.17) means that there is no sliding mode in the boundary layer of width $\dfrac{\sup|\mathbf{HW}|.\bar{v}}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}}$, so strictly speaking, it is a pseudo-sliding mode.

Remark 4.7: Validity of Theorem 4.6 under No Disturbance

If there is no disturbance, we have

$$\sup|\mathbf{HW}|.\bar{v} = 0$$

then by Eq.(4.17.c), the perturbation control vanishes

$$\mathbf{K}_p = \mathbf{0}$$

and the control function in Eq.(4.17) becomes exactly the same as in Eq.(4.8) for the case without perturbation.

4.4.2.2. Robust Linear SMC under Parametric Uncertainties

Lemma 4.1: Computation of Special Eigenvalues

For any $\mathbf{H} \in \Re^{1\times n}$ and any $\mathbf{M} \in \Re^{n\times n}$, we have

$$\mathbf{Eig}\left[\mathbf{H}^T\mathbf{HM} + \left(\mathbf{H}^T\mathbf{HM}\right)^T\right] = \left[\alpha \pm \sqrt{\beta}, \quad \mathbf{0}\right] \qquad (4.18)$$

where $$\alpha = \mathbf{HMH}^T, \quad \beta = \mathbf{HH}^T(\mathbf{HM})(\mathbf{HM})^T \qquad (4.18.a)$$

Proof:

We will prove by induction. By the definition of eigenvalues $\mathrm{Eig}(\mathbf{P}) = \left\{\lambda_i \middle| |\lambda_i\mathbf{I} - \mathbf{P}| = 0\right\}$, we can prove that Eq.(4.18) is true for $n = 2$ . . By induction, Eq.(4.18) is true up to n.

Q.E.D.

We propose the following theorem for both *matched and un-matched* uncertain dynamical systems.

Theorem 4.7: Robust Linear SMC Design under Parametric Uncertainty

Consider a linear system under both matched and unmatched parametric uncertainty

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}.\mathbf{x} + \tilde{\mathbf{B}}.u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u$$

and a hyperplane

$$s = \mathbf{H}.\mathbf{x}$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B}$$

with

$$\mathbf{x} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B}, \Delta\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re$$

then under Assumption 4.1, there exists a sliding margin $\delta > 0$ for a robust SMC to be determined as

$$u = -\mathbf{K}.\mathbf{x}, \quad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p \tag{4.19}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.19.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\mathbf{H}.\delta \tag{4.19.b}$$

- perturbation control

$$\mathbf{K}_p = \left[\frac{|\mathbf{H}|.\left(\Delta\mathbf{A} + \Delta\mathbf{B}.|\mathbf{K}_e + \mathbf{K}_r|\right)}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\mathrm{sgn}(\mathbf{HB})}\right] \otimes \mathrm{sgn}(\mathbf{H}) \tag{4.19.c}$$

with

$$\mathbf{K}_e, \mathbf{K}_r, \mathbf{K}_p \in \Re^{1\times n}, \quad \delta \in \Re_{(+)}, \quad |\mathbf{M}| = \left[\left|m_{ij}\right|\right], \quad \left[p_{ij}\right] \otimes \left[q_{ij}\right] = \left[p_{ij}.q_{ij}\right]$$

Proof

We have

$$\dot{s} = \mathbf{H}\left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \mathbf{H}\left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u = \mathbf{HAx} + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} + \mathbf{HB}u + \mathbf{H}.\Delta\tilde{\mathbf{B}}.u$$

by Eq.(4.19), we obtain

$$\dot{s} = \mathbf{HAx} - \mathbf{HAx} - \delta s + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} - \mathbf{H}.\Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right).\mathbf{x} - \mathbf{H}\tilde{\mathbf{B}}\mathbf{K}_p\mathbf{x}$$

so

$$s\dot{s} = -\delta s^2 - s\mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]\mathbf{x} = -\delta s^2 - \mathbf{x}^T\mathbf{H}^T\mathbf{HM}.\mathbf{x} \tag{4.20}$$

where

$$\mathbf{M} = \tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}} \tag{4.21}$$

The quadratic form requires a symmetric matrix, Eq.(4.20) can be read as

$$s\dot{s} = -\delta s^2 - \tfrac{1}{2}\mathbf{x}^T\left[\mathbf{H}^T\mathbf{HM} + \left(\mathbf{H}^T\mathbf{HM}\right)^T\right]\mathbf{x} \tag{4.22}$$

The sliding mode $s\dot{s} < 0$ is satisfied if the quadratic term is non-negative, that is the matrix is positive-semi-definite, so all eigenvalues must be non-negative. By Lemma 4.1, this can be achieved if both sum and product of 2 eigenvalues are non-negative, that is

$$2\alpha \ge 0 \tag{4.23.a}$$

$$\alpha^2 - \beta \ge 0 \tag{4.23.b}$$

We will use the first condition in Eq.(4.23.a) to compute the perturbation control and the second condition will be used to check.

(a)

Eq.(4.18.a) can be written as

$$\alpha = \mathbf{H}.\mathbf{M}.\mathbf{H}^T \tag{4.24}$$

In view of Eqs.(4.19.c) and (4.21), we get

$$\alpha = \mathbf{HMH}^T = \mathbf{H\tilde{B}}\left\{\left[\frac{|\mathbf{H}|.\left(\Delta\mathbf{A} + \Delta\mathbf{B}.|\mathbf{K}_e + \mathbf{K}_r|\right)}{\inf\left|\mathbf{H\tilde{B}}\right|.\mathrm{sign}(\mathbf{HB})}\right] \otimes \mathrm{sgn}(\mathbf{H})\right\}.\mathbf{H}^T - \mathbf{H}.\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right].\mathbf{H}^T$$

or

$$\alpha = \frac{|\mathbf{HB}|}{\inf\left|\mathbf{H\tilde{B}}\right|}|\mathbf{H}|.\left(\Delta\mathbf{A} + \Delta\mathbf{B}.|\mathbf{K}_e + \mathbf{K}_r|\right).|\mathbf{H}|^T - \mathbf{H}.\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right].\mathbf{H}^T$$

since $\frac{|\mathbf{HB}|}{\inf\left|\mathbf{H\tilde{B}}\right|} > 1$ by Assumption 4.1, so

$$\alpha > |\mathbf{H}|.\left(\Delta\mathbf{A} + \Delta\mathbf{B}.|\mathbf{K}_e + \mathbf{K}_r|\right).|\mathbf{H}|^T - \mathbf{H}.\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right].\mathbf{H}^T > 0$$

(b)

By Eq.(4.18.a), Eq.(4.23.b) can be written as

$$\alpha^2 = \left(\mathbf{LH}^T\right)^2, \quad \beta = \|\mathbf{H}\|^2\|\mathbf{L}\|^2 \tag{4.25}$$

where

$$\mathbf{L} = \mathbf{HM} \tag{4.25.a}$$

By the Schwartz's Inequality

$$\left|\mathbf{LH}^T\right| \le \|\mathbf{L}\|.\|\mathbf{H}\| \quad \Rightarrow \quad \alpha \le \sqrt{\beta} \tag{4.26}$$

so there is a negative eigenvalue. However, both sides of Eq.(4.26) are the same order, so the negative eigenvalue can be expected too small compared to the positive one, *ie*. the quadratic term in Eq.(4.20) will contribute negative value to the sliding condition. In addition, the first term in Eq.(4.20) will even make more negative.

Q.E.D.

Remark 4.8: Robust Linear SMC Design under No Uncertainty

If there is no uncertainty, we have

$$\tilde{\mathbf{A}} = \mathbf{A}, \quad \tilde{\mathbf{B}} = \mathbf{B}, \quad \Delta\tilde{\mathbf{A}} = \mathbf{0}, \quad \Delta\tilde{\mathbf{B}} = \mathbf{0}$$

then by Eq.(4.26.c), the perturbation control vanishes, and the control function in Eq.(4.26) is exactly the same as in Eq.(4.8) for the case without perturbation.

Corollary 4.4: Robust Linear SMC Design under Reduced Parametric Variation

If

$$\Delta\mathbf{B} = \mathbf{0}$$

then the linear SMC in Theorem 4.7 becomes

$$u = -\mathbf{K}.\mathbf{x} = -\left(\mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p\right)\mathbf{x} \tag{4.27}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \tag{4.27.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\mathbf{H}.\delta \tag{4.27.b}$$

- perturbation control

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\left(|\mathbf{H}|.\Delta\mathbf{A}\right)\otimes \operatorname{sgn}(\mathbf{H}) \tag{4.27.c}$$

Remark 4.9: Equivalent Control in Robust SMC Design

The first component in Eq.(4.27) is termed "equivalent control" due to the same the argument in Remark 4.4, that is the control function reduces to the equivalent control once in the sliding mode.

4.4.2.3. Robust Linear SMC under Matched Parametric Uncertainties

Under uncertainties, all what can be guaranteed is that if the sliding-eigenvalues are Hurwitz, then so are the system-eigenvalues, *ie*. the closed-loop system is stable (Theorem 3.1). Under the matching condition, we have very interesting characteristics by the following theorem.

Theorem 4.8: Robust Linear SMC Design under Matched Uncertainty

If a matched uncertain system has all elements of **H** of the same polarity, then the system-eigenvalues at the upper boundary for are

$$\lambda_{C,2} = \{-\delta, \quad \lambda_H\} \tag{4.28}$$

Proof

From Eq.(4.3), a matched uncertain system takes the form

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \mathbf{B}.u$$

where

$$\Delta\tilde{\mathbf{A}} = \mathbf{B}.\mathbf{v}$$

From Eq.(4.19.c) with $\Delta\tilde{\mathbf{B}} = \mathbf{0}$ and with $h_i's$ of the same polarity, without loss of generality, assume the positive polarity

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\mathbf{H}.\Delta\mathbf{A} = \pm(\mathbf{HB})^{-1}\mathbf{H}.\mathbf{B}.\mathbf{v} = \pm\mathbf{v} \tag{4.29}$$

Thus the system-eigenvalues at the upper boundary, $\mathbf{A} = \mathbf{A} + \Delta\mathbf{A}$, are

$$\lambda_{C2} = \operatorname{eig}\left[\mathbf{A} + \Delta\mathbf{A} - \mathbf{B}.\left(\mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p\right)\right] = \operatorname{eig}\left[\mathbf{A} \pm \mathbf{B}.\mathbf{v} - \mathbf{B}.\left(\mathbf{K}_{er} + \mathbf{v}\right)\right] = \operatorname{eig}\left[\mathbf{A} - \mathbf{B}.\mathbf{K}_{er}\right], \quad \mathbf{K}_{er} = \mathbf{K}_e + \mathbf{K}_r$$

Recall that, under no perturbation we have

$$\mathrm{eig}\left(\mathbf{A}-\mathbf{B}.\mathbf{K}_{er}\right)=\{-\delta, \quad \lambda_H\}$$

so

$$\lambda_{C2}=\mathrm{eig}\left[\mathbf{A}-\mathbf{B}.\mathbf{K}_{er}\right]=\{-\delta, \quad \lambda_H\}$$

Q.E.D.

Theorem 4.8 has shown that system eigenvalues of an uncertain dynamical system at upper boundary $(\mathbf{A}+\Delta\mathbf{A})$ are equal to those of a deterministic dynamical system. We propose the following design rule to move these eigenvalues inside the bounds by shifting down the nominal values of the system matrix.

Proposition 4.1: Robust Linear SMC Design Rule

Theorem 4.8 is valid for matched uncertain systems and **H** has all element the same polarity. In general, choosing the nominal system matrices as

$$\mathbf{A}\to\mathbf{A}+\tfrac{1}{2}\Delta\mathbf{A}, \qquad \mathbf{B}\to\mathbf{B}+\tfrac{1}{2}\Delta\mathbf{B}$$

in designing a hyperplane may produce a more robust controller.

Remark 4.10: Advantage of Robust Design Rule

An advantage from the proposition above is system-eigenvalues of the nominal model are $\{-\delta, \quad \lambda_H\}$ for matched uncertain systems and **H** has all element the same polarity

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4.4.2.4. Robust Linear SMC under Un-matched Uncertainty and Disturbances

From Theorem 4.6 and 4.7, we propose the following corollary to design a robust linear SMC for systems under unmatched uncertainty and disturbances

Corollary 4.5: Robust Linear SMC Design under Un-matched Uncertainty and Disturbance

Consider a linear system under unmatched parametric uncertainty and external disturbances

$$\dot{\mathbf{x}}=\tilde{\mathbf{A}}.\mathbf{x}+\tilde{\mathbf{B}}.u+\mathbf{W}.v=\left(\mathbf{A}+\Delta\tilde{\mathbf{A}}\right).\mathbf{x}+\left(\mathbf{B}+\Delta\tilde{\mathbf{B}}\right).u+\mathbf{W}.v \tag{4.30}$$

and a hyperplane

$$s=\mathbf{H}.\mathbf{x}$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right|\le\Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right|\le\Delta\mathbf{B}, \qquad |v|\le\bar{v}$$

with

$$\mathbf{x}\in\Re^{n\times1}, \quad \mathbf{A},\Delta\mathbf{A}\in\Re^{n\times n}, \quad \mathbf{B},\Delta\mathbf{B}\in\Re^{n\times1}, \quad \mathbf{H}\in\Re^{1\times n}, \quad u,s\in\Re$$

then, under Assumption 4.1, a robust linear SMC function is determined by

$$\underline{\underline{u=-\mathbf{K}.\mathbf{x}, \quad \mathbf{K}=\mathbf{K}_e+\mathbf{K}_r+\mathbf{K}_p}} \tag{4.31}$$

where

- equivalent control

$$\mathbf{K}_e=(\mathbf{HB})^{-1}\mathbf{H}.\mathbf{A} \tag{4.31.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\mathbf{H}.\delta \tag{4.31.b}$$

- perturbation control

$$\mathbf{K}_p = \mathbf{K}_u + \mathbf{K}_d \tag{4.31.c}$$

* uncertainty control

$$\mathbf{K}_u = \left[\frac{|\mathbf{H}|.\left(\Delta\mathbf{A} + \Delta\mathbf{B}.|\mathbf{K}_e + \mathbf{K}_r|\right)}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\mathrm{sgn}(\mathbf{HB})}\right].\otimes\,\mathrm{sgn}(\mathbf{H}) \tag{4.31.d}$$

* disturbance control

$$\mathbf{K}_d = (\mathbf{HB})^{-1}\delta_p\mathbf{H}.\left(\sup|\mathbf{HW}|.\bar{v}\right) \tag{4.31.e}$$

if the following disturbance condition is satisfied

$$|s| > \frac{\sup|\mathbf{HW}|.\bar{v}}{\delta + \delta_p \sup|\mathbf{HW}|.\bar{v}} \tag{4.31.f}$$

with

$$\mathbf{K}_e, \mathbf{K}_r, \mathbf{K}_p, \mathbf{K}_u, \mathbf{K}_d \in \Re^{1\times n}, \quad \delta, \delta_p \in \Re_{(+)}, \quad |\mathbf{M}| = \left[|m_{ij}|\right]$$

Proof

Eqs.(4.31.a), (4.31.b) and (4.31.d) can be determined by Theorem 4.7. Since the uncertainty control tackles uncertainty, Eq.(4.30) can be read as

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u + \mathbf{W}.v \tag{4.32}$$

so Eq.(4.31.e) can be determined from Eq.(4.32) using Theorem 4.6.

Q.E.D.

4.4.2.5. Alternative Formula for Robust Linear SMC

Alternative to Theorem 4.7, under Assumption 4.1, we propose a theorem for a robust linear SMC. We first present the following lemma

Lemma 4.2: Alternative Computation of Special Eigenvalues

For any $\mathbf{V} \in \Re^{1\times n}$ and any $\mathbf{M} \in \Re^{n\times n}$, we have

$$\mathrm{Eig}\left[\mathbf{V}^T\mathbf{V}.\mathbf{M}\right] = \left[\mathbf{VMV}^T, \quad 0, \quad \cdots, \quad 0\right] \tag{4.33}$$

Proof:

We will prove by induction. By the definition of eigenvalues $\mathrm{Eig}(\mathbf{P}) = \left\{\lambda_i \middle| |\lambda_i\mathbf{I} - \mathbf{P}| = 0\right\}$, we can prove that Eq.(4.33) is true for $n = 3$.

Let

$$\mathbf{P}_3 = \mathbf{V}^T\mathbf{VM} = \begin{bmatrix} v_1\sum_{i=1}^{3} v_i m_{i1} & v_1\sum_{i=1}^{3} v_i m_{i2} & v_1\sum_{i=1}^{3} v_i m_{i3} \\ v_2\sum_{i=1}^{3} v_i m_{i1} & v_2\sum_{i=1}^{3} v_i m_{i2} & v_2\sum_{i=1}^{3} v_i m_{i3} \\ v_3\sum_{i=1}^{3} v_i m_{i1} & v_3\sum_{i=1}^{3} v_i m_{i2} & v_3\sum_{i=1}^{3} v_i m_{i3} \end{bmatrix} \tag{4.34}$$

so

$$\left|\lambda \mathbf{I}-\mathbf{V}^{T}\mathbf{V}\mathbf{M}\right|=\lambda^{2}\left(\lambda-\sum_{i=1}^{3}\sum_{j=1}^{3}v_{i}v_{j}m_{ij}\right) \tag{4.35}$$

Assume that Eq.(4.33) is true up to $n-1$. Note that the non-zero eigenvalue in Eq.(4.35) is equal to the sum of diagonal elements in Eq.(4.34). So we can prove that Eq.(4.33) is also true for n by using some properties of the trace of a matrix as follows

$$\mathrm{tr}\left(\mathbf{P}_{n}\right)=\sum_{i=1}^{n}p_{ii}$$

$$\mathrm{tr}\left(\mathbf{P}_{n}\right)=\sum_{i=1}^{n}\lambda_{i}, \qquad \mathrm{Eig}\left(\mathbf{P}_{n}\right)=\left\{\lambda_{i}\right\}$$

$$\mathrm{tr}\left(\mathbf{P}_{a}+\mathbf{P}_{b}\right)=\mathrm{tr}\left(\mathbf{P}_{a}\right)+\mathrm{tr}\left(\mathbf{P}_{b}\right)$$

where

$$\mathbf{P}_{n}=\mathbf{V}^{T}\mathbf{V}\mathbf{M}=\begin{bmatrix}v_{1}\\ v_{2}\\ \vdots\\ v_{n}\end{bmatrix}\begin{bmatrix}v_{1} & v_{2} & \cdots & v_{n}\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1n}\\ m_{21} & m_{22} & \cdots & m_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n1} & m_{n2} & \cdots & m_{nn}\end{bmatrix}=\begin{bmatrix}v_{1}v_{1} & v_{1}v_{2} & \cdots & v_{1}v_{n}\\ v_{2}v_{1} & v_{2}v_{2} & \cdots & v_{2}v_{n}\\ \vdots & \vdots & \ddots & \vdots\\ v_{n}v_{1} & v_{n}v_{2} & \cdots & v_{n}v_{n}\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1n}\\ m_{21} & m_{22} & \cdots & m_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n1} & m_{n2} & \cdots & m_{nn}\end{bmatrix}$$

or

$$\mathbf{P}_{n}=\begin{bmatrix}v_{1}\sum_{i=1}^{n}v_{i}m_{i1} & v_{1}\sum_{i=1}^{n}v_{i}m_{i2} & \cdots & v_{1}\sum_{i=1}^{n}v_{i}m_{in}\\ v_{2}\sum_{i=1}^{n}v_{i}m_{i1} & v_{2}\sum_{i=1}^{n}v_{i}m_{i2} & \cdots & v_{2}\sum_{i=1}^{n}v_{i}m_{in}\\ \vdots & \vdots & \ddots & \vdots\\ v_{n}\sum_{i=1}^{n}v_{i}m_{i1} & v_{n}\sum_{i=1}^{n}v_{i}m_{i2} & \cdots & v_{n}\sum_{i=1}^{n}v_{i}m_{in}\end{bmatrix}$$

and

$$\mathbf{P}_{n}=\mathbf{P}_{a}+\mathbf{P}_{b}, \qquad \mathbf{P}_{a}=\mathbf{P}_{n-1}, \qquad \mathbf{P}_{b}=\begin{bmatrix}0 & \cdots & 0 & v_{1}\sum_{i=1}^{n}v_{i}m_{in}\\ \vdots & \ddots & \vdots & v_{2}\sum_{i=1}^{n}v_{i}m_{in}\\ 0 & \cdots & 0 & \vdots\\ v_{n}\sum_{i=1}^{n}v_{i}m_{i1} & v_{n}\sum_{i=1}^{n}v_{i}m_{i2} & \cdots & v_{n}\sum_{i=1}^{n}v_{i}m_{in}\end{bmatrix}$$

Q.E.D.

We then have the following theorem

Theorem 4.9: Alternative Robust Linear SMC Design

Consider the following system

$$\dot{\mathbf{x}}=\tilde{\mathbf{A}}\mathbf{x}+\tilde{\mathbf{B}}u=\left(\mathbf{A}+\Delta\tilde{\mathbf{A}}\right)\mathbf{x}+\left(\mathbf{B}+\Delta\tilde{\mathbf{B}}\right)u, \qquad \left|\Delta\tilde{\mathbf{A}}\right|\le\Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right|\le\Delta\mathbf{B} \tag{4.36}$$

and

$$s=\mathbf{H}\mathbf{x}$$

then under Assumption 4.1, a robust SMC can be determined by

$$u=-\mathbf{K}\mathbf{x}, \qquad \mathbf{K}=\mathbf{K}_{e}+\mathbf{K}_{r}+\mathbf{K}_{p} \tag{4.37}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA} \qquad (4.37.a)$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\delta\mathbf{H} \qquad (4.37.b)$$

- perturbation control

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\gamma\ \mathbf{H}, \qquad \gamma = \frac{\left|\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}.\left|\mathbf{K}_e + \mathbf{K}_r\right|\right)\mathbf{H}^T\right|}{\left(1 - |\mathbf{HB}|^{-1}|\mathbf{H}|.\Delta\mathbf{B}\right)\mathbf{HH}^T} \qquad (4.37.c)$$

Proof:

From Eq.(4.37), we have

$$\dot{s} = \mathbf{H}\tilde{\mathbf{A}}\mathbf{x} + \mathbf{H}\tilde{\mathbf{B}}u = \mathbf{HAx} + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} + \mathbf{HB}u + \mathbf{H}.\Delta\tilde{\mathbf{B}}.u$$

by Eq.(4.37), we obtain

$$\dot{s} = \mathbf{HAx} - \mathbf{HAx} - \delta s + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x} - \mathbf{H}.\Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right).\mathbf{x} - \mathbf{H}\tilde{\mathbf{B}}\mathbf{K}_p\mathbf{x}$$

so

$$s\dot{s} = -\delta s^2 - s\mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]\mathbf{x} = -\delta s^2 - \mathbf{x}^T\mathbf{H}^T\mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]\mathbf{x}$$

thus $s\dot{s} < 0$ only if $\mathbf{H}^T\mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]$ is semi-positive definite. If so required, Lemma 4.2 yields

$$\lambda = \mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]\mathbf{H}^T > 0 \qquad (4.38)$$

In view of Eq.(4.19.c), Eq.(4.38) can be read as

$$\lambda = \mathbf{H}.\left[\tilde{\mathbf{B}}\mathbf{K}_p + \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right) - \Delta\tilde{\mathbf{A}}\right]\mathbf{H}^T = \mathbf{H}\tilde{\mathbf{B}}\mathbf{K}_p\mathbf{H}^T - \mathbf{H}\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right]\mathbf{H}^T$$

$$\lambda = \frac{\left|\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}.\left|\mathbf{K}_e + \mathbf{K}_r\right|\right)\mathbf{H}^T\right|}{\left(1 - |\mathbf{HB}|^{-1}|\mathbf{H}|.\Delta\mathbf{B}\right)} - \mathbf{H}\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right]\mathbf{H}^T$$

or

$$\lambda > \left|\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}.\left|\mathbf{K}_e + \mathbf{K}_r\right|\right)\mathbf{H}^T\right| - \mathbf{H}\left[\Delta\tilde{\mathbf{A}} - \Delta\tilde{\mathbf{B}}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right]\mathbf{H}^T$$

since Eq.(4.6) in Assumption 4.1 gives

$$\left|\mathbf{H}.\Delta\tilde{\mathbf{B}}\right| < |\mathbf{HB}| \quad \Rightarrow \quad 0 < |\mathbf{H}|\Delta\mathbf{B} < |\mathbf{HB}| \quad \Rightarrow \quad 0 < |\mathbf{HB}|^{-1}|\mathbf{H}|\Delta\mathbf{B} < 1 \quad \Rightarrow \quad 0 < 1 - |\mathbf{HB}|^{-1}|\mathbf{H}|.\Delta\mathbf{B} < 1$$

so Eq.(4.38) can be achieved.

Q.E.D.

If $\Delta\tilde{\mathbf{B}} = \mathbf{0}$, we have the following corollary

Corollary 4.6: Alternative Robust Linear SMC Design under Reduced Parametric Variation

Consider the following system

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \mathbf{B}u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \mathbf{B}u, \qquad \left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}$$

and

$$s = \mathbf{Hx}$$

then a robust SMC can be determined by

$$u = -\mathbf{Kx}, \qquad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{HA}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\delta\mathbf{H}$$

- perturbation control

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\gamma\mathbf{H}, \qquad \gamma = \frac{\left|\mathbf{H}.\Delta\mathbf{A}.\mathbf{H}^T\right|}{\mathbf{H}\mathbf{H}^T}$$

4.5. ROBUST INTEGRAL SMC DESIGN

A SMC is based on a state-space model, an I-action may be required to eliminate a steady-state error. We have the following corollary to design a robust integral SMC using Theorem 4.3 or 4.9.

Theorem 4.10: Robust Integral Linear SMC Design under Uncertainty

Consider an uncertain dynamical linear system system

$$\begin{cases} \dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}u = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u, & \left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \quad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B} \\ y = \mathbf{C}\mathbf{x} \end{cases}$$

and its augmented-order system with a reference input of r

$$\dot{\mathbf{x}}_i = \tilde{\mathbf{A}}_i.\mathbf{x}_i + \tilde{\mathbf{B}}_i.u \tag{4.39}$$

where

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{0}_{n\times 1} & \tilde{\mathbf{A}} \end{bmatrix}, \quad \tilde{\mathbf{B}}_i = \begin{bmatrix} 0 \\ \tilde{\mathbf{B}} \end{bmatrix} \tag{4.39.a}$$

then a robust integral SMC with a hyperplane $\mathbf{H}_i$ can be determined by

$$\underline{\underline{u = u_* + \tilde{u}_i}} \tag{4.40}$$

where

$$u_* = \left(\mathbf{H}_i\mathbf{B}_i\right)^{-1} h_{i,1}\, r, \quad \mathbf{H}_i = \begin{bmatrix} h_{i,1} & \cdots & h_{i,n+1} \end{bmatrix} \tag{4.40.a}$$

and

$\tilde{u}_i$ is a robust SMC for Eq.(4.39) and can be determined by Theorem 4.3 or 4.7.

with

$$\mathbf{x}, \mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{A} \in \Re^{n\times n}, \quad \mathbf{C} \in \Re^{1\times n}, \quad u, u_*, \tilde{u}_i, y, r \in \Re: \text{ sliding margin.}$$

Proof

Let

$$x_0 = \int_0^t (y - r)\, dt \quad \Rightarrow \quad \dot{x}_0 = y - r$$

then

$$\begin{bmatrix} \dot{x}_0 \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} y - r \\ \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}u \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{x} \\ \tilde{\mathbf{A}}\mathbf{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\mathbf{B}}u \end{bmatrix} + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{0}_{n\times 1} & \tilde{\mathbf{A}} \end{bmatrix}\begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\mathbf{B}} \end{bmatrix}.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r$$

from Eq.(4.39.a), we have

$$\dot{\mathbf{x}}_i = \mathbf{A}_i.\mathbf{x}_i + \mathbf{B}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r \tag{4.41}$$

since the uncertain component of u can tackle any uncertainty $\Delta\tilde{\mathbf{B}}_i$, Eq.(4.41) is still hold for $\Delta\tilde{\mathbf{B}}_i = \mathbf{0}$ so it can be read as

$$\dot{\mathbf{x}}_i = \tilde{\mathbf{A}}_i.\tilde{\mathbf{x}} + \tilde{\mathbf{B}}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \tilde{\mathbf{A}}_i.\tilde{\mathbf{x}} + \mathbf{B}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r \tag{4.41.a}$$

thus

$$\dot{s}_i = \mathbf{H}_i\dot{\mathbf{x}}_i = \mathbf{H}_i\tilde{\mathbf{A}}_i\tilde{\mathbf{x}}_i + \mathbf{H}_i\mathbf{B}_i u + \mathbf{H}_i.\begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \mathbf{H}_i\tilde{\mathbf{A}}_i\tilde{\mathbf{x}}_i + \mathbf{H}_i\mathbf{B}_i u - h_{i,1}\, r$$

by Eq.(4.40.a), we have

$$\dot{s}_i = \mathbf{H}_i\tilde{\mathbf{A}}_i\tilde{\mathbf{x}}_i + \mathbf{H}_i\mathbf{B}_i u - \mathbf{H}_i\mathbf{B}_i u_* = \mathbf{H}_i\tilde{\mathbf{A}}_i\tilde{\mathbf{x}}_i + \mathbf{H}_i\mathbf{B}_i\tilde{u}_i \tag{4.42}$$

since $\tilde{u}_i$ is a robust SMC for Eq.(4.39), the sliding condition $\tilde{s}\dot{\tilde{s}} \le 0$ is satisfied using Eq.(4.42)

Q.E.D.

4.6. A NEW ROBUST LINEAR SLIDING-MODE OBSERVER DESIGN

As in the state-space observer design where an the response of an observer is 3 to 10 faster than that of a controller, to design a robust linear Sliding-Mode Observer (SMO), we propose the following theorem

Theorem 4.11: Robust Linear Sliding-Mode Observer Design under Uncertainty

Consider the following uncertain dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}u = \left(\mathbf{A}\mathbf{x} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u \\ y = \tilde{\mathbf{C}}\mathbf{x} = \left(\mathbf{C} + \Delta\tilde{\mathbf{C}}\right)\mathbf{x} \end{cases} \tag{4.43}$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B}$$

if the system is observable, then a robust linear SMO can be found from

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}}), \qquad \hat{\mathbf{y}} = \tilde{\mathbf{C}}\hat{\mathbf{x}} \tag{4.44}$$

where

$$\underline{\underline{\mathbf{L} = \mathbf{L}_e + \mathbf{L}_r + \mathbf{L}_p}} \tag{4.44.a}$$

with

$$\mathbf{L}_e = \mathbf{A}\mathbf{H}_o^{/}\left(\mathbf{C}\mathbf{H}_o^{/}\right)^{-1} \tag{4.44.b}$$

$$\mathbf{L}_r = \delta_o\mathbf{H}_o^{/}\left(\mathbf{C}\mathbf{H}_o^{/}\right)^{-1}, \qquad \delta_o = (3..10)\times\delta \tag{4.44.c}$$

$$\mathbf{L}_p = \operatorname{sgn}(\mathbf{H}_o)\otimes\frac{\left(\Delta\mathbf{A}^{/} + \left|\mathbf{L}_e + \mathbf{L}_r\right|.\Delta\mathbf{C}\right).\left|\mathbf{H}_o^{/}\right|}{\inf\left|\mathbf{H}_o\tilde{\mathbf{C}}^{/}\right|.\operatorname{sgn}\left(\mathbf{H}_o\mathbf{C}^{/}\right)} \tag{4.44.d}$$

or the alternative

$$\mathbf{L}_p = \gamma_o\mathbf{H}_o^{/}\left(\mathbf{C}\mathbf{H}_o^{/}\right)^{-1}, \qquad \gamma_o = \frac{\left|\mathbf{H}_o\right|.\left[\Delta\mathbf{A}^{/} + \Delta\mathbf{C}^{/}.\left|\mathbf{L}_e + \mathbf{L}_r\right|\right].\left|\mathbf{H}_o^{/}\right|}{\left[1 - \left|\left(\mathbf{H}_o\mathbf{C}^{/}\right)^{-1}\right|.\left|\mathbf{H}_o\right|.\Delta\mathbf{C}^{/}\right].\mathbf{H}_o\mathbf{H}_o^{/}} \tag{4.44.e}$$

and

$\mathbf{H}_o$ is determined by the eigenvalue allocation with the observer hyperplane-eigenvalues

$$\lambda_{Ho} = (3..10)\times\lambda_{Hc} \quad\Rightarrow\quad \mathbf{H}_o = \operatorname{hyper}\left(\mathbf{A}^{/}, \mathbf{C}^{/}, \lambda_{H_o}\right)$$

$$\hat{\mathbf{x}} \in \Re^{n\times 1}, \quad \mathbf{y} \in \Re^{p\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B}, \Delta\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{C}, \Delta\mathbf{C} \in \Re^{p\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re$$

Proof

Since **L** will be determined under uncertainty, Eq.(4.44) can be perturbed to read as

$$\dot{\hat{\mathbf{x}}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\hat{\mathbf{x}} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u + \mathbf{L}.\left(\mathbf{y} - \hat{\mathbf{y}}\right) = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \tilde{\mathbf{B}}u + \mathbf{L}.\left(\mathbf{y} - \hat{\mathbf{y}}\right)$$

then an error equation $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$ can be determined by

$$\dot{\mathbf{e}} = \dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \tilde{\mathbf{B}}u + \mathbf{L}.\left(\mathbf{y} - \hat{\mathbf{y}}\right) - \tilde{\mathbf{A}}.\mathbf{x} - \tilde{\mathbf{B}}.u = \left(\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}}\right).\mathbf{e} \tag{4.45}$$

To determine **L**, we will find a function v to nullify the error by the following equation

$$\dot{\mathbf{e}} = \tilde{\mathbf{A}}^{/}\mathbf{e} + \tilde{\mathbf{C}}^{/}v \tag{4.46}$$

By Theorem 4.7, we can find

$$v = -\mathbf{L}^{/}\ \mathbf{e} \tag{4.47}$$

where $\mathbf{L}^{/}$ is determined by Eqs.(4.44), so Eq.(4.46) can be read

$$\dot{\mathbf{e}} = \tilde{\mathbf{A}}^{/}\mathbf{e} - \tilde{\mathbf{C}}^{/}\tilde{\mathbf{L}}^{/}\mathbf{e} = \left(\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}}\right)^{/}\mathbf{e} \tag{4.48}$$

since $eig(\mathbf{M}) = eig\left(\mathbf{M}^{/}\right)$, we have the following mapping to complete the proof

$$\tilde{\mathbf{A}} \to \tilde{\mathbf{A}}^{/}, \quad \tilde{\mathbf{B}} \to \tilde{\mathbf{C}}^{/}, \quad \mathbf{H} \to \mathbf{H}_o, \quad \mathbf{K} \to \mathbf{L}^{/} \tag{4.49}$$

so Eqs.(4.44) have been proved.

Q.E.D.

4.7. ANALYSIS OF MATCHED AND UNMATCHED UNCERTAINTIES

We will analyze matched and unmatched uncertainties in transfer function and state-space models.

4.7.1. Transfer Function Model

Consider a system

$$G(s) = \frac{b_0\left(b_1 s + 1\right)}{s^2 + a_1 s + a_0} \tag{4.50}$$

then the minimal realization is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} \end{cases} \tag{4.51}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1, & b_1 \end{bmatrix}, \quad \mathbf{D} = 0 \tag{4.51a}$$

We want to find a transformation **T**

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}} \tag{4.52}$$

such that

$$\mathbf{CT} = \begin{bmatrix} 1, & 0 \end{bmatrix} \tag{4.53}$$

thus

$$\mathbf{T} = \begin{bmatrix} 1 & b_1 \\ 0 & -1 \end{bmatrix} \Leftrightarrow \mathbf{T}^{-1} = \begin{bmatrix} 1 & b_1 \\ 0 & -1 \end{bmatrix} \tag{4.54}$$

then

$$y = \mathbf{C}\mathbf{x} = \mathbf{C}\left(\mathbf{T}\tilde{\mathbf{x}}\right) = (\mathbf{CT}).\tilde{\mathbf{x}} = \tilde{\mathbf{C}}.\tilde{\mathbf{x}} = \tilde{x}_1 \tag{4.55}$$

and Eq.(4.51) becomes

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}u \\ y = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{cases} \tag{4.56}$$

where

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} -b_1a_0 & -1 + b_1a_1 - b_1^2a_0 \\ a_0 & -a_1 + b_1a_0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} b_1b_0 \\ -b_0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{T} = \begin{bmatrix} 1, & 0 \end{bmatrix} \tag{4.57}$$

(a) If $b_1 = 0$, then Eqs.(4.50) & (4.57) gives

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \tag{4.58}$$

and

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & -1 \\ a_0 & -a_1 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ -b_0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1, & 0 \end{bmatrix} \tag{4.59}$$

Thus, if all parameters are uncertain, we have *matched* uncertain system.

(b) If $b_1 \neq 0$, assume that

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} . \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = -k_1\tilde{x}_1 - k_2\tilde{x}_2 \tag{4.60}$$

then Eqs.(4.56) & (4.57) give

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \hat{\mathbf{A}}\tilde{\mathbf{x}} + \hat{\mathbf{B}}u \\ y = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{cases} \tag{4.61}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} -b_1a_0 - k_1b_1b_0 & -1 + b_1a_1 - b_1^2a_0 - k_2b_1b_0 \\ a_0 & -a_1 + b_1a_0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ -b_0 \end{bmatrix} \tag{4.62}$$

thus it is *unmatched* uncertain system.

We can generalize the above analysis to have the following proposition

Proposition 4.1: Uncertainty in Transfer Function Model

In general, we have

- A *matched* uncertain system has the model

$$G(s) = \frac{\tilde{b}_0}{s^n + \tilde{a}_{n-1}s^{n-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0} \tag{4.63}$$

where the minimal realization can be used.

- An unmatched uncertain system has the model in Eq.(4.63), but its minimal realization cannot be applied due to inaccessibility of system states, or it has the model

$$G(s) = \frac{\tilde{b}_{n-1}s^{n-1} + \cdots + \tilde{b}_1 s + \tilde{b}_0}{s^n + \tilde{a}_{n-1}s^{n-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0} \tag{4.64}$$

where $\tilde{a}_i, \tilde{b}_i$ are uncertain.

4.7.2. State-Space Model

We will analyze an uncertain state-space model and consider if we can convert an unmatched uncertain system into a matched uncertain system.

4.7.2.1. Analysis of Uncertain State-Space Model

Consider an uncertain system

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u = \mathbf{A}\mathbf{x} + \Delta\tilde{\mathbf{A}}.\,\mathbf{x} + \mathbf{B}u + \Delta\tilde{\mathbf{B}}.\,u \tag{4.65}$$

Let

$$\Delta\tilde{\mathbf{A}} = \mathbf{W}\mathbf{z} \tag{4.66}$$

and

$$\Delta\tilde{\mathbf{B}}.\,u = \mathbf{B}.\,v_1 \tag{4.67}$$

where

$$\Delta\tilde{\mathbf{A}} \in \Re^{n\times n},\ \ \mathbf{W} \in \Re^{n\times 1}$$

and $\forall \mathbf{z} \in \Re^{1\times n}$, $\forall v_1 \in \Re$ are *uncertainties*. Then Eq.(4.63) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{W}\mathbf{z}.\,\mathbf{x} + \mathbf{B}u + \mathbf{B}v_1 \tag{4.68}$$

Since **z** is an uncertainty, let

$$\mathbf{z}\mathbf{x} = v_2 \tag{4.69}$$

where v_2 is also an uncertainty; then Eq.(4.68) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(u + v_1) + \mathbf{W}v_2 \tag{4.70}$$

Therefore,

- for a matched uncertain system, we have

$$\mathbf{W} = \mathbf{B} \tag{4.71}$$

so Eq.(4.70) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(u + v_1 + v_2) = \mathbf{A}\mathbf{x} + \mathbf{B}(u + v) \tag{4.72}$$

where $v = v_1 + v_2$, thus we have a disturbance to *u*.

- for an unmatched uncertain system, we have both disturbances to *u* and to $\dot{\mathbf{x}}$. However, to determine the bound of v_2, we need to know the bound of **x**, *ie.* the ranges of x_i, $i = 1, 2, \cdots, n$. Eq.(4.69) reveals that the bound of v_2 depends on the bound of x_i's which are proportional to the reference, for example, while it is not the case for a normal disturbance. For example, consider an unmatched uncertain system

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ a_1 & 0 & 1 \\ a_2 & a_3 & a_4 \end{bmatrix},\quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix},\quad \Delta\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 \\ \Delta\tilde{a}_1 & 0 & 0 \\ \Delta\tilde{a}_2 & 0 & 0 \end{bmatrix},\quad \Delta\tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \Delta\tilde{b}_1 \end{bmatrix} \tag{4.73}$$

then choosing

$$\mathbf{W} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\quad \mathbf{z} = \begin{bmatrix} 0 & \Delta\tilde{a}_1 & \Delta\tilde{a}_2 \end{bmatrix} \tag{4.74}$$

gives, by Eq.(4.69)

$$v_2 = \mathbf{zx} = \begin{bmatrix} 0 & \Delta\tilde{a}_1 & \Delta\tilde{a}_2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 . \Delta\tilde{a}_1 + x_3 . \Delta\tilde{a}_2 \tag{4.75}$$

so

$$\bar{v}_2 = \bar{x}_2 . \Delta a_1 + \bar{x}_3 . \Delta a_2 \tag{4.76}$$

where

$$|v_2| \le \bar{v}_2 , \quad |x_1| \le \bar{x}_1 , \quad |x_2| \le \bar{x}_2 , \quad |\Delta\tilde{a}_1| \le \Delta a_1 , \quad |\Delta\tilde{a}_2| \le \Delta a_2$$

4.7.2.2. Conversion of Uncertain State-Space Model

Consider the state-space model

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \;\Rightarrow\; s\mathbf{X} = \mathbf{AX} + \mathbf{B}U \;\Rightarrow\; (s\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{B}U \tag{4.77}$$

then

$$y = \mathbf{Cx} \;\Rightarrow\; Y = \mathbf{CX} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U \;\Rightarrow\; G(s) = \frac{Y}{U} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{4.78}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \tilde{a}_1 & 0 & 1 \\ \tilde{a}_2 & a_3 & a_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \tilde{b}_1 \end{bmatrix} \tag{4.79}$$

thus

$$G(s) = \frac{\tilde{b}_1}{s^3 - a_4 s^2 - (a_3 + \tilde{a}_1)s + \tilde{a}_1 a_4 - \tilde{a}_2} \tag{4.80}$$

so

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (\tilde{a}_2 - \tilde{a}_1 a_4) & (a_3 + \tilde{a}_1) & a_4 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \tilde{b}_1 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{4.81}$$

We will find a transformation **T** such that

$$\tilde{\mathbf{x}} = \mathbf{Tx} \tag{4.82}$$

then

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}u \tag{4.83}$$

where

$$\tilde{\mathbf{A}} = \mathbf{TAT}^{-1}, \quad \tilde{\mathbf{B}} = \mathbf{TB} \tag{4.84}$$

thus solving the equation

$$\begin{cases} \tilde{\mathbf{A}}\mathbf{T} = \mathbf{TA} \\ \tilde{\mathbf{B}} = \mathbf{TB} \end{cases} \tag{4.85}$$

to obtain

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tilde{a}_1 & 0 & 1 \end{bmatrix} \tag{4.86}$$

Due to uncertainty, $\tilde{a}_1$ is unknown, so is the transformation **T**, thus it is *impossible* to transform the *unmatched* uncertain model in Eq.(4.79) by Eq.(4.82) into the *matched* uncertain system in Eq.(4.81). Therefore, the above analysis yields the following proposition

Proposition 4.2: Conversion of Unmatched to Matched Uncertain System

In general, it is impossible to find a (constant) transformation from an unmatched to matched uncertain system. However, if all system states are inaccessible except the output and an observer is required, then there *may* exist a conversion via transfer function to transform an unmatched to matched uncertain system (Example 4.8).

4.8. NUMERICAL EXAMPLES

In order to see the elimination of the chattering problem, the discontinuous SMC (VSS) will be included. The TanH VSS control will be included to compare with the linear SMC since both are able to eliminate the chattering problem, but the former is a pseudo-sliding mode while the latter is a true sliding mode. So we sumarize the results as follows

Remark 4.11: Summary of Robust SMC Designs

In the numerical examples below, and in this work generally, hyperplane eigenvalues will be chosen at the same unique value for simplicity. Different multiple values may be attempted to compromise between the response speed and overshoot.

- Nominal model can be determined using proposition 4.1 in designing a hyperplane (Chapter 2)
- Sliding margin δ is chosen on the basis of Proposition 3.1;
- New Robust Discontinuous SMC functions are computed by Theorem 4.3 where the switching function is given by Eq.(3.5). Saturate, Unitvector and TanH VSS control functions are given by Eqs.(3.6) to (3.8), respectively, based on Proposition 3.2;
- New Robust Linear (Continuous) SMC functions are computed by Corollary 4.5 deduced from Theorems 4.6 and 4.7 for a general case under uncertainty and disturbance where a special case without perturbation (uncertainty and/or disturbance) will make the corresponding control component(s) vanish;
- Robust Integral SMC functions are computed by Theorem 4.10;
- New robust sliding-mode observers are computed by Theorem 4.11

4.8.1. Example 4.1: No Perturbation

Consider a linear system (Sivaramakrishnan *et al.* 1984)

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u, \qquad \mathbf{A} = \begin{bmatrix} -0.05 & 6 & 0 & 0 \\ 0 & -3.333 & 3.333 & 0 \\ -5.208 & 0 & -12.5 & -12.5 \\ 0.6 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 12.5 \\ 0 \end{bmatrix}$$

4.8.1.1. Original Design

From Sivaramakrishnan *et al.* 1984, we have the original control function as

$$u = \psi_1 x_1 + \psi_2 x_2 + \psi_3 x_3 + \psi_4 x_4$$

where

$$\psi_1 = \begin{cases} -6, & \text{if } s.x_1 > 0 \\ +6, & \text{if } s.x_1 < 0 \end{cases} \qquad \psi_2 = \begin{cases} -6, & \text{if } s.x_2 > 0 \\ +6, & \text{if } s.x_2 < 0 \end{cases}$$

$$\psi_3 = \begin{cases} -2, & \text{if } s.x_3 > 0 \\ +2, & \text{if } s.x_3 < 0 \end{cases} \qquad \psi_4 = \begin{cases} 0, & \text{if } s.x_3 > 0 \\ 0, & \text{if } s.x_3 < 0 \end{cases}$$

and

$$s = \mathbf{H}.\mathbf{x}, \qquad \mathbf{H} = \begin{bmatrix} 5.155, & 4.385, & 1 & 16 \end{bmatrix}$$

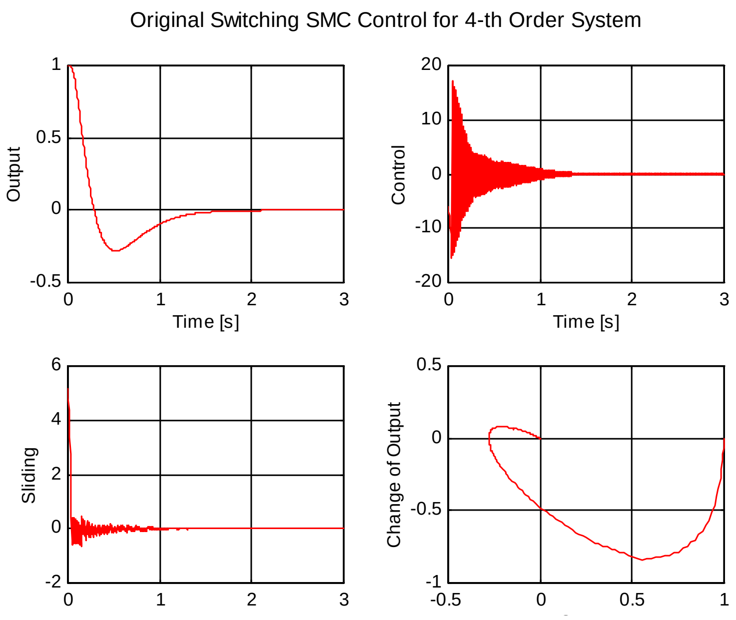


Fig. 4.1: Original Switching SMC Control for Linear System in Example 4.1.

4.8.1.2. New Design

To compare to the original design, choose $\lambda_H = \begin{bmatrix} -6, & -6, & -6 \end{bmatrix}$, then

$$\mathbf{H} = \begin{bmatrix} 0.4285, & 0.3508, & 0.0800, & 1.4401 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 0.4260 & 1.4014 & 0.1694 & -1 \end{bmatrix}$$

and the sliding-eigenvalues

$$\lambda_S = \lambda_H = \begin{bmatrix} -6, & -6, & -6 \end{bmatrix}$$

- TanH SMC (Continuous Pseudo-SMC)

  Choose $k_s = 20$ and $\delta = 10$, then Theorem 4.3 yields

$$\mathbf{K}_r = [4.2845 \quad 3.5084 \quad 0.8 \quad 14.4014]$$

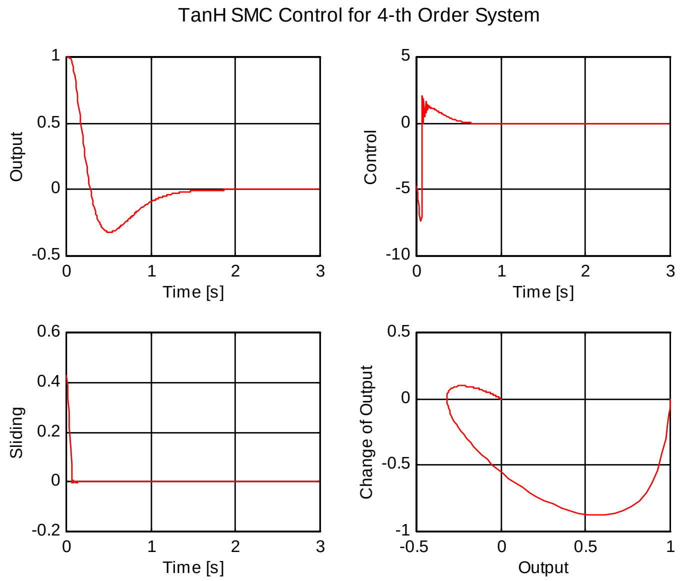


Fig. 4.2: TanH SMC Control for Linear System without Perturbation in Example 4.1.

- Linear SMC

  Choose $\delta = 18$, then Corollary 4.5 yields

$$\mathbf{K}_r = [7.7122, \quad 6.3152, \quad 1.4400, \quad 25.9226]$$

thus

$$\mathbf{K} = [8.1382, \quad 7.7165, \quad 1.6094, \quad 24.9226]$$

and the system-eigenvalues

$$\lambda_C = [-18, \quad -6, \quad -6, \quad -6]$$

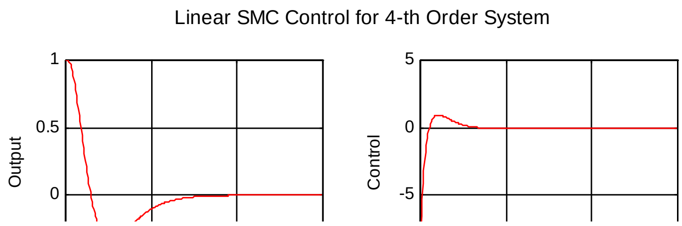


Fig. 4.3: Linear SMC for Linear System without Perturbation in Example 4.1.

- Performance

The performance of the TanH SMC control is comparable to that of the linear SMC. Note that the former is a pseudo-sliding mode, whereas the latter is a true sliding mode. Moreover, the latter is a linear control so the linear control theory can be applied to obtain more insight such as the closed-loop system eigenvalues.

The following are modified models to include perturbations where their hyperplane equations, and equivalent controls are exactly the same as in the Example 4.1 above, while reaching control is modified for a proper reaching dynamics due to the design rule (Proposition 3.1)

4.8.2. Example 4.2: Matched Uncertainties

Consider a matched uncertain linear system in Coleman *et al.* 1994

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2.5 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

Choose $\lambda_H = \begin{bmatrix} -2 & -2 \end{bmatrix}$, then

$$\mathbf{H} = \begin{bmatrix} 0.5254 & 0.5293 & 0.1333 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 0 & 0.4587 & 0.2293 \end{bmatrix}$$

due to the matching condition, the sliding-eigenvalues are absolutely unchanged under matched uncertainties and equal to the hyperplane-eigenvalues:

$$\boldsymbol{\lambda}_{S1} = \boldsymbol{\lambda}_{S2} = \boldsymbol{\lambda}_H = \begin{bmatrix} -2 & -2 \end{bmatrix}$$

We have

$$\mathbf{K}_p = \begin{bmatrix} 0 & 0.2 & 0.1 \end{bmatrix}$$

choose $\delta = 3$, then Theorem 4.3 yields

$$\mathbf{K}_r = \begin{bmatrix} 1.5761 & 1.5880 & 0.4 \end{bmatrix}$$

Robust Switching SMC Control for Matched Uncertain 3-rd Order System

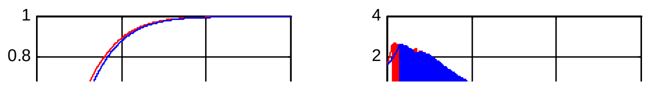


Fig. 4.4: Robust SMC Control for Matched Uncertain System in Example 4.2.

- TanH SMC (Continuous Pseudo-SMC)

  Choose $k_s = 25$

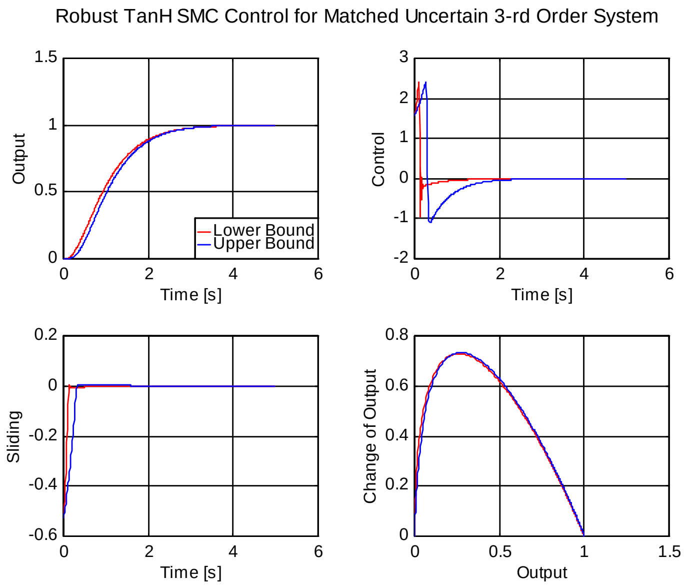


Fig. 4.5: Robust TanH SMC Control for Matched Uncertain System in Example 4.2.

- Linear SMC

  We have

$$\mathbf{K}_p = \begin{bmatrix} 2.6268 & 3.3054 & 0.9960 \end{bmatrix}$$

Choose $\delta = 5$, then Corollary 4.5 yields

$$\mathbf{K}_r = \begin{bmatrix} 2.6268 & 2.6467 & 0.6667 \end{bmatrix}$$

thus

$$\mathbf{K} = \begin{bmatrix} 5.2536 & 6.4107 & 1.8920 \end{bmatrix}$$

but the system-eigenvalues always vary with the uncertainties

$$\lambda_{C1} = \begin{bmatrix} -1.41 & -2 & -27.98 \end{bmatrix}, \quad \lambda_{C2} = \begin{bmatrix} -1.69 & -2 & -7.77 \end{bmatrix}$$

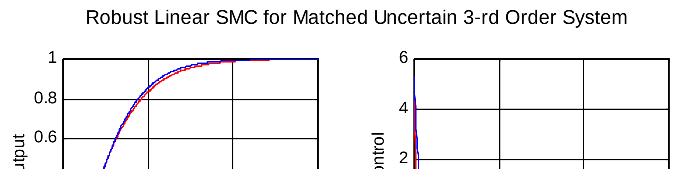


Fig. 4.6: Robust Linear SMC for Matched Uncertain Dynamical System in Example 4.2.

- Performance

The performance of the TanH SMC control is comparable to that of the linear SMC. Note that the former is a pseudo-sliding mode, whereas the latter is a true sliding mode. Moreover, the latter is a linear control so the linear control theory can be applied to obtain more insight such as the closed-loop system eigenvalues.

Remark 4.12: Efficiency of Robust Linear SMC Design Rule (Proposition 4.1)

Without using Proposition 4.1, we have the following system eigenvalues

$$\lambda_{C1} = \begin{bmatrix} -1.44 & -2 & -20.47 \end{bmatrix}, \qquad \lambda_{C2} = \begin{bmatrix} -2 & -2 & -5 \end{bmatrix}$$

which is less robust than the above eigenvalues using the proposition. Also note that λ_{C2} verifies Theorem 4.6.

4.8.3. Example 4.3: Un-matched Uncertainty

Consider an un-matched uncertain linear system in Chen *et al.* 1989 where disturbance is deferred to the next example

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 1 \\ 8 & -7 & -8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.3 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$$

Choose $\lambda_H = \begin{bmatrix} -5 & -5 \end{bmatrix}$, then

$$\mathbf{H} = \begin{bmatrix} 20.6897 & 6.8966 & 0.6897 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 41.7241 & 15.8621 & 1.3793 \end{bmatrix}$$

due to un-matched uncertainties, the sliding-eigenvalues are changed under un-matched uncertainties, but they are still Hurwitz:

$$\lambda_{S1} = \begin{bmatrix} -5 \pm j\,1.73 \end{bmatrix}, \qquad \lambda_{S2} = \begin{bmatrix} -4 & -6 \end{bmatrix}$$

- Robust Switching SMC (Discontinuous SMC)

We have

$$\mathbf{K}_p = \begin{bmatrix} 25 & 0 & 0 \end{bmatrix}$$

Choose $\delta = 2$, then Theorem 4.3 yields

$$\mathbf{K}_r = \begin{bmatrix} 41.3793 & 13.7931 & 1.3793 \end{bmatrix}$$

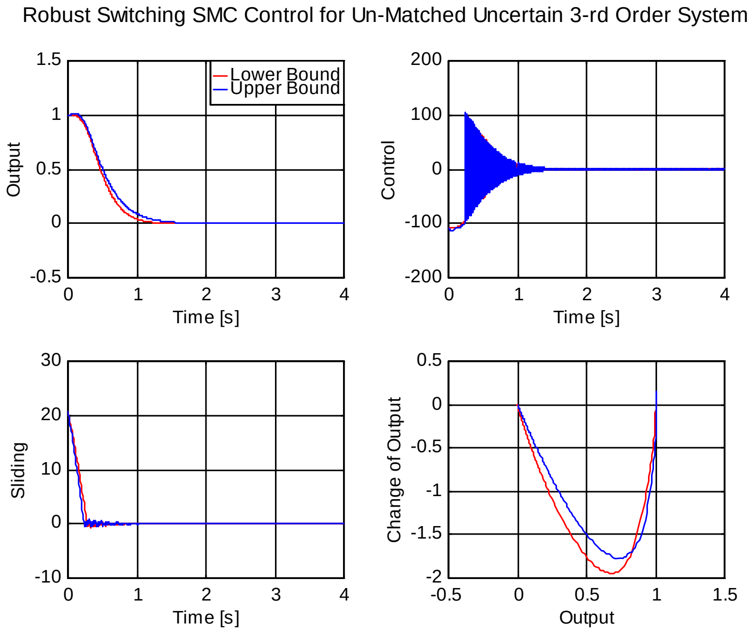


Fig. 4.7: Robust SMC Control for Un-Matched Uncertain System in Example 4.3.

- Robust TanH SMC (Continuous Pseudo-SMC)

  Choose $k_s = 2$

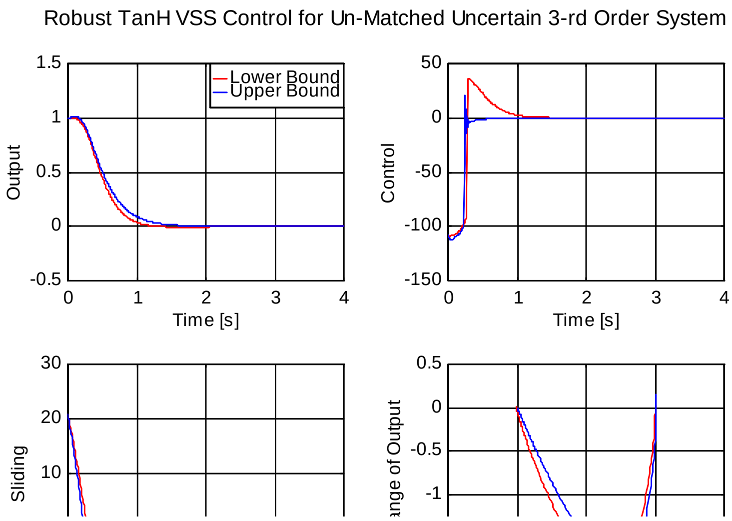


Fig. 4.8: Robust TanH SMC Control for Un-Matched Uncertain Dynamical System in Example 4.3.

- Robust Linear SMC

  We have

$$\mathbf{K}_p = \begin{bmatrix} 99.5862 & 25.4483 & 2.4828 \end{bmatrix}$$

Choose $\delta = 10$, then Corollary 4.5 yields

$$\mathbf{K}_r = \begin{bmatrix} 206.8966 & 68.9655 & 6.8966 \end{bmatrix}$$

thus $$\mathbf{K} = \begin{bmatrix} 348.2069 & 110.2759 & 10.7586 \end{bmatrix}$$

but the system-eigenvalues always vary with the uncertainties

$$\lambda_{C1} = \begin{bmatrix} -10.64 & -4.06 \pm j3.53 \end{bmatrix}, \quad \lambda_{C2} = \begin{bmatrix} -15.29 & -4.96 \pm j1.03 \end{bmatrix}$$

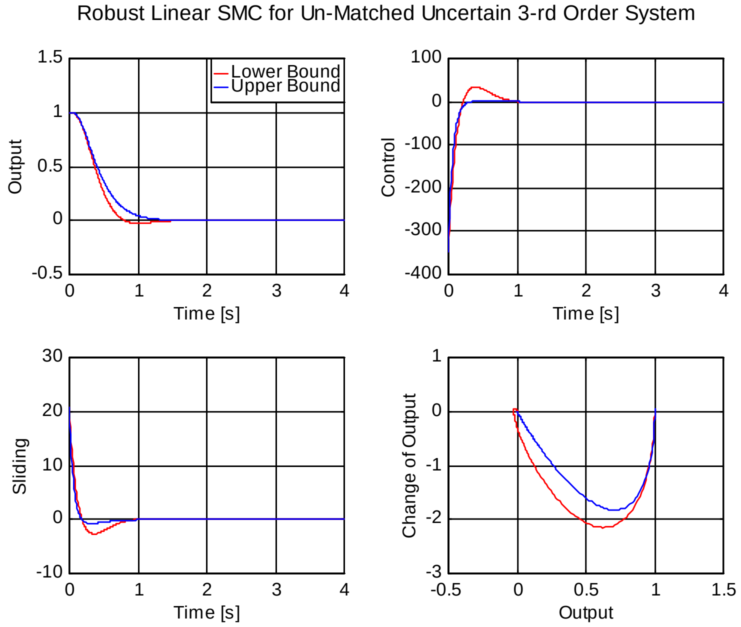


Fig. 4.9: Robust Linear SMC for Un-Matched Uncertain System in Example 4.3.

Remark 4.13: Temporary Violation of Sliding Violation

There is a region where $s < 0$ and $\dot{s} < 0$ so $s.s > 0$, that is the sliding condition $s.\dot{s} < 0$ is not satisfied and $|s|$ is getting larger instead of smaller as required. However, the proof of Theorem 4.7 has shown that when $|s|$ increases to a certain magnitude, the sliding condition is satisfied and thus $|s|$ reduces to 0.

- Performance

The performance of the TanH SMC control is comparable to that of the linear SMC. Note that the former is a pseudo-sliding mode, whereas the latter is a true sliding mode. Moreover, the latter is a linear control so the linear control theory can be applied to obtain more insight such as the closed-loop system eigenvalues.

4.8.4. Example 4.4: Un-matched Uncertainty and External Disturbance

Consider an un-matched uncertain linear system under disturbance in Chen *et al.* 1989

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u + \mathbf{W}v, \quad |v| \le 1$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 1 \\ 8 & -7 & -8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.3 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with the disturbance is chosen as follows for simulation

$$v = \sin(50t)$$

All design parameters are the same as in Example 4.3 with an additional disturbance control.

- Robust Switching SMC

  Eq.(4.7.c) yields $K_{p0} = 30$

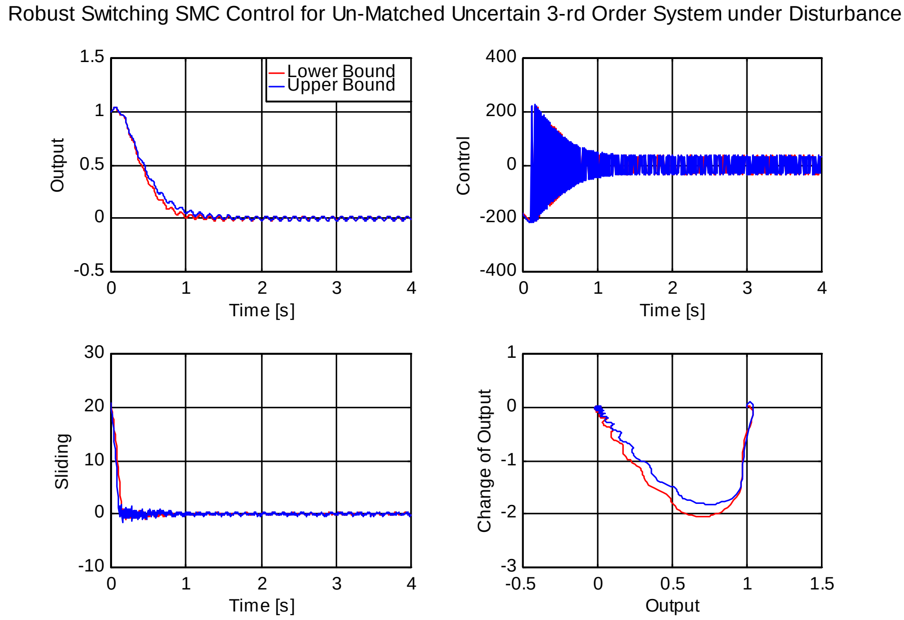


Fig. 4.10: Robust SMC Control under Unmatched Uncertainties and Disturbances in Example 4.3.

- Robust TanH SMC

  Choose $k_s = 2$

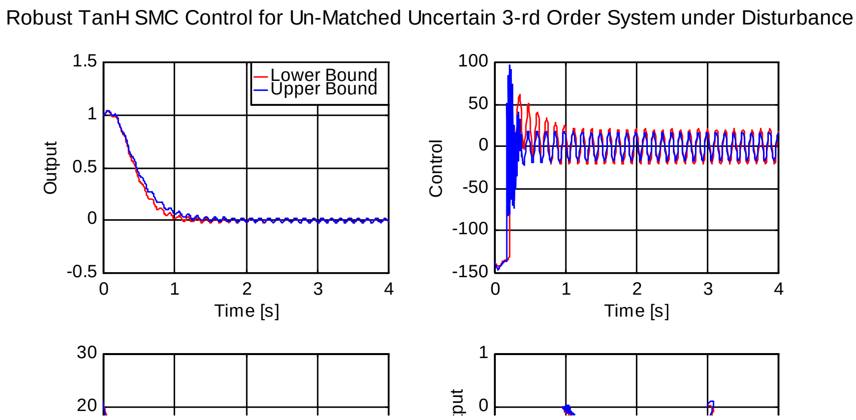


Fig. 4.11: Robust TanH SMC Control under Unmatched Uncertainties and Disturbances in Example 4.3.

- Robust Linear SMC

Choose $\delta_p = 0.2$, then Corollary 4.5 yields

$$\mathbf{K}_d = [124.1379 \quad 41.3793 \quad 4.1379]$$

thus

$$\mathbf{K} = [472.3448 \quad 151.6552 \quad 14.8966]$$

but the system-eigenvalues always vary with the uncertainties

$$\lambda_{C1} = [-13.74 \quad -4.58 \pm j3.14], \qquad \lambda_{C2} = [-21.83 \quad -5 \pm j0.51]$$

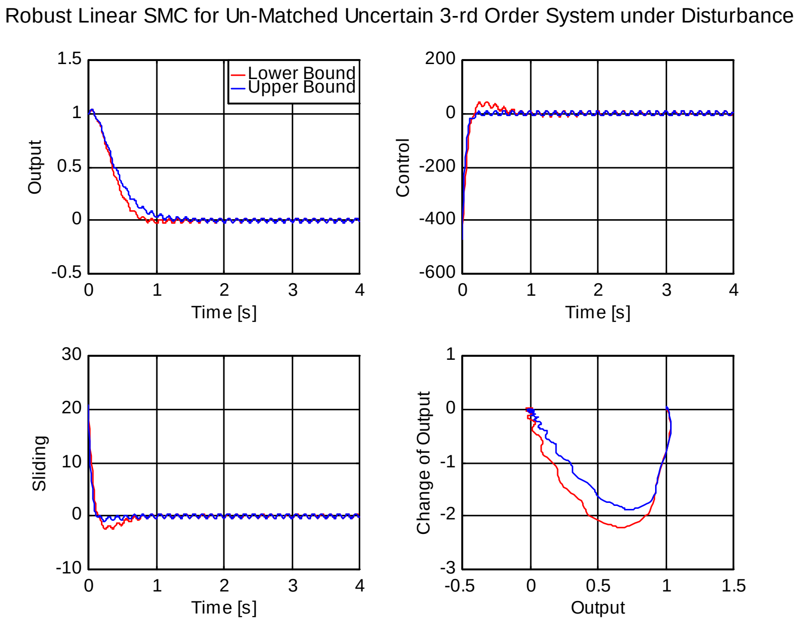


Fig. 4.12: Robust Linear SMC under Unmatched Uncertainties and Disturbances in Example 4.3.

Note the violation of the sliding condition (Remark 4.13).

- Performance

The performance of the TanH SMC control is comparable to that of the linear SMC one. Note that the former is a pseudo-sliding mode, whereas the latter is a true sliding mode. Moreover, the latter is a linear control so the linear control theory can be applied to obtain more insight such as the closed-loop system eigenvalues.

4.8.5. Example 4.5: Linear SMC for Nonlinear Systems

Consider a nonlinear from Zhou *et al.* 1992

$$\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}.u$$

where

$$\mathbf{f} = \begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1) \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 1 + \sqrt{|x_1|} \end{bmatrix}$$

since any physical system is bounded-input and bounded-output, we can have the following assumption

$$|x_1| \le 1$$

the above system can be rewritten as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = ax_1 + 2bx_2 + c.u \end{cases}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u, \qquad \mathbf{A} = \begin{bmatrix} 0, & 1 \\ a & 2b \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

where

$$-1 \le a \le 2, \quad -1 \le b \le 1, \quad 1 \le c \le 2$$

then

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u = (\mathbf{A} + \Delta\mathbf{A}).\mathbf{x} + (\mathbf{B} + \Delta\mathbf{B}).u, \qquad \mathbf{A} = \begin{bmatrix} 0, & 1 \\ 0.5 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \qquad \Delta\mathbf{A} = \begin{bmatrix} 0, & 0 \\ 1.5 & 2 \end{bmatrix}, \qquad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

Choose $\lambda_H = [-2]$, then

$$\mathbf{H} = [2, \quad 1]$$

thus

$$\mathbf{K}_e = [-0.2 \quad 0.8]$$

choose $\delta = 4$, hence Corollary 4.5 yields

$$\mathbf{K}_{rp} = [11 \quad 7.2]$$

so a linear SMC is

$$u = -\mathbf{K}.\mathbf{x}, \qquad \mathbf{K} = [10, \quad 8]$$

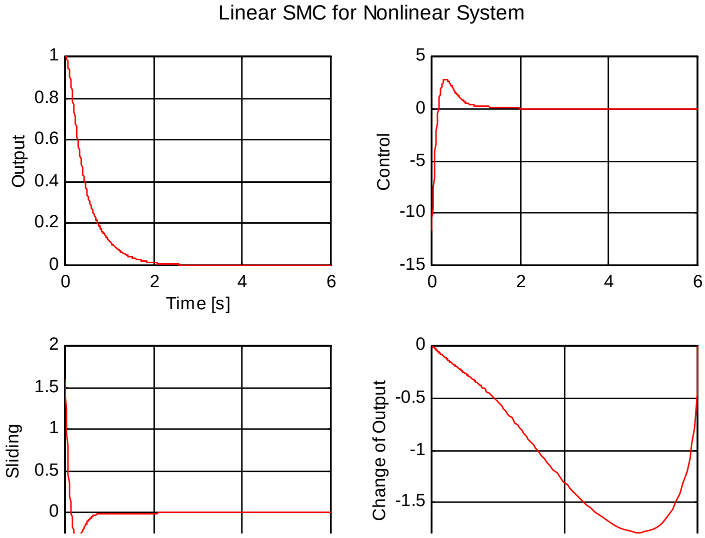


Fig. 4.13: Linear SMC for Nonlinear System in Example 4.5.

Note the violation of the sliding condition (Remark 4.13).

4.8.6. Example 4.6: Robust Integral SMC under Uncertainty and Disturbance

Consider the following system under uncertainty from Lin *et al.* 1992 where a disturbance is included to illustrate the design

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \mathbf{B}\,u + \mathbf{W}\,v, \qquad v = 10\,\sin(500t)$$

where

$$\mathbf{A} = \begin{bmatrix} -25 & -3 \\ -13 & 84 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -48 \\ -58 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 48 & 0 \\ 58 & 0 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v| \le 10$$

4.8.6.1. Original Design

Using the linear quadratic optimal control to minimize the perturbation effect, an original control has been

$$u = -1.078\, x_1 + 4.819\, x_2$$

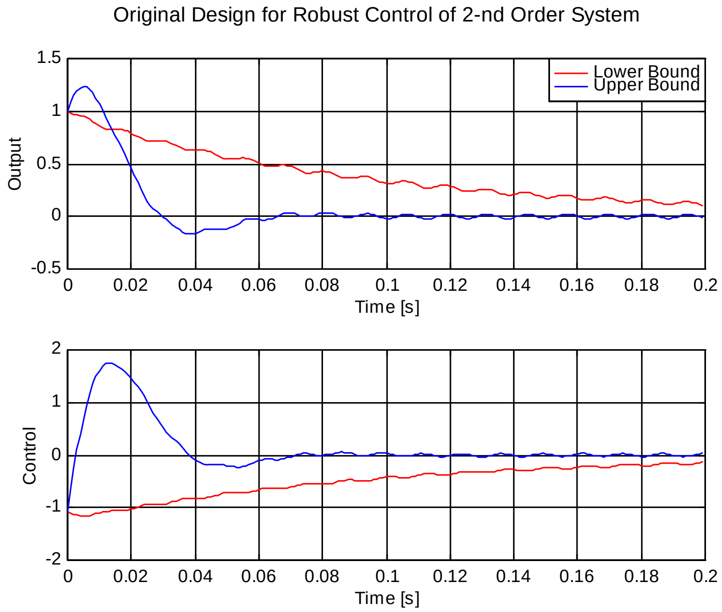


Fig. 4.14: Original Control under Uncertainty and Disturbance in Example 4.6.

If a unit step input is used

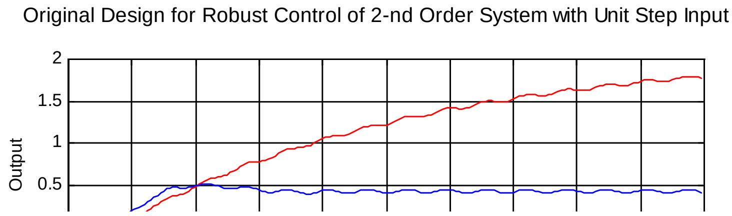


Fig. 4.15: Original Control under Uncertainty and Disturbance with Unit Step Input in Example *4.6: Steady-State Error*

4.8.6.2. New Robust Integral SMC Design

Choose $\lambda_H = [-50]$, then

$$\tilde{\mathbf{H}} = [0.5944, \quad 0.0234, \quad -0.0366]$$

thus

$$\mathbf{K}_e = [0, \quad 0.9856, \quad -3.1433]$$

- Robust Integral SMC (Discontinuous SMC)

Choose $\delta = 3$, , then Theorem 4.10 yields

$$\mathbf{K}_{rp} = \mathbf{K}_r + \mathbf{K}_p = [8.9105 \quad 1.3505 \quad 0.5487]$$

and

$$k_0 = 5.9439$$

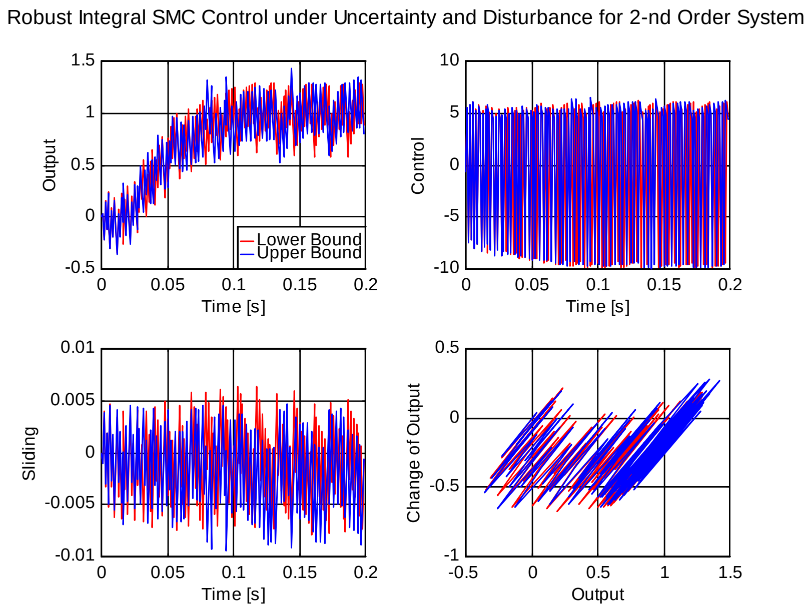


Fig. 4.16: Robust Integral SMC Control under Uncertainty and Disturbance with Unit Step Input in Example 4.6: *No Steady-State Error*

- Robust Integral TanH SMC (Continuous Pseudo-SMC), $k_s = 250$

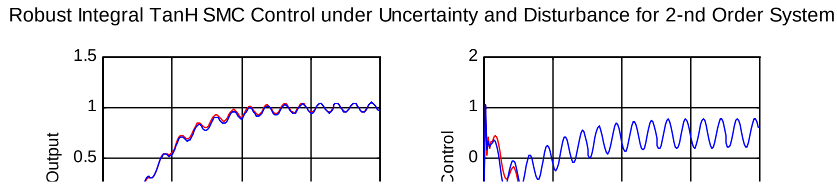


Fig. 4.17: Robust Integral TanH SMC Control under Uncertainty and Disturbance with Unit Step Input in Example 4.6: *No Steady-State Error*

- Robust Integral Linear SMC

Choose $\delta = 400$, , then Theorem 4.10 yields

$$\mathbf{K} = \begin{bmatrix} 237.7556, & 9.8353, & -17.7775 \end{bmatrix}$$

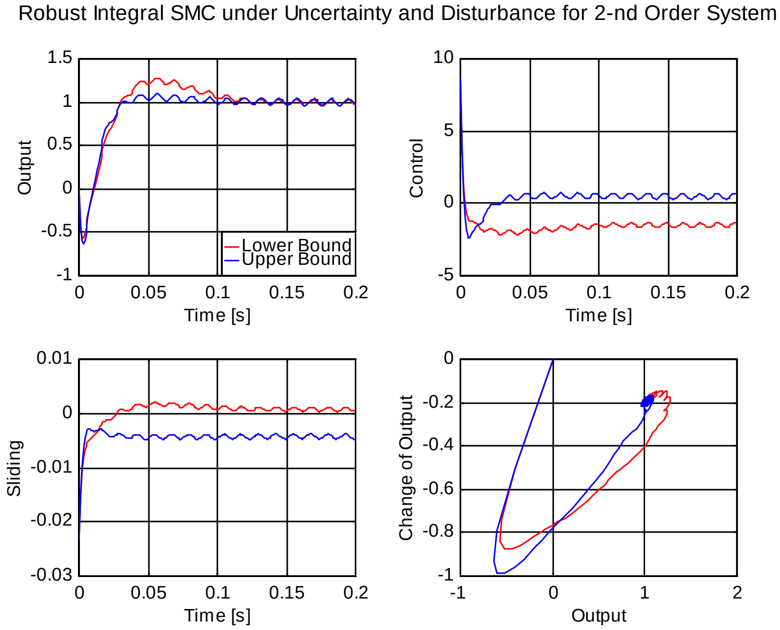


Fig. 4.18: Robust Integral Linear SMC under Uncertainty and Disturbance with Unit Step Input in Example 4.6: *No Steady-State Error*

4.8.7. Example 4.7: Robust Sliding-Mode Observer under Un-Matched Uncertainties

Consider an un-matched uncertain linear system under external disturbance in Example 4.4 above, using the observer dynamics 2 times faster than that of the robust linear SMC, , then Theorem 4.11 yields

$$\mathbf{L} = \begin{bmatrix} 71.2090 \\ 241.3402 \\ -110.0076 \end{bmatrix}, \quad \mathbf{A}_e = \mathbf{A} - \mathbf{LC} = \begin{bmatrix} -71.2090 & 1 & 0 \\ -237.3402 & 0 & 1 \\ 118.0076 & -7 & -8 \end{bmatrix}$$

This observer can be used for both robust discontinuous and continuous sliding-mode controllers in Example 4.4. To compare with Linear SMC, the TanH SMC Control will be used rather than the Switching SMC Control.

Remark 4.14: Performance of Pseudo-SMC

In the linear SMC which is a true SMC, the sliding variable *s* asymptotically goes to zero, but it is never equal to zero. Strictly speaking, the linear SMC is also a pseudo-SMC, that is why the performance of linear SMC is similar to that of pseudo-SMC such as TanH-SMC in all simulations above.

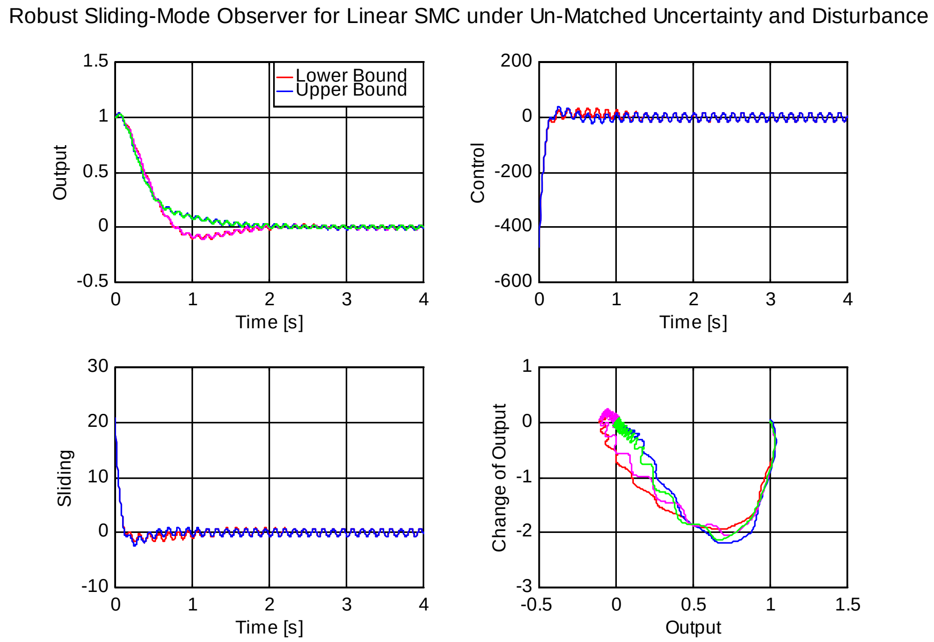


Fig. 4.19: Robust Sliding-Mode Observer for Linear Sliding-Mode Controller in Example 4.7.

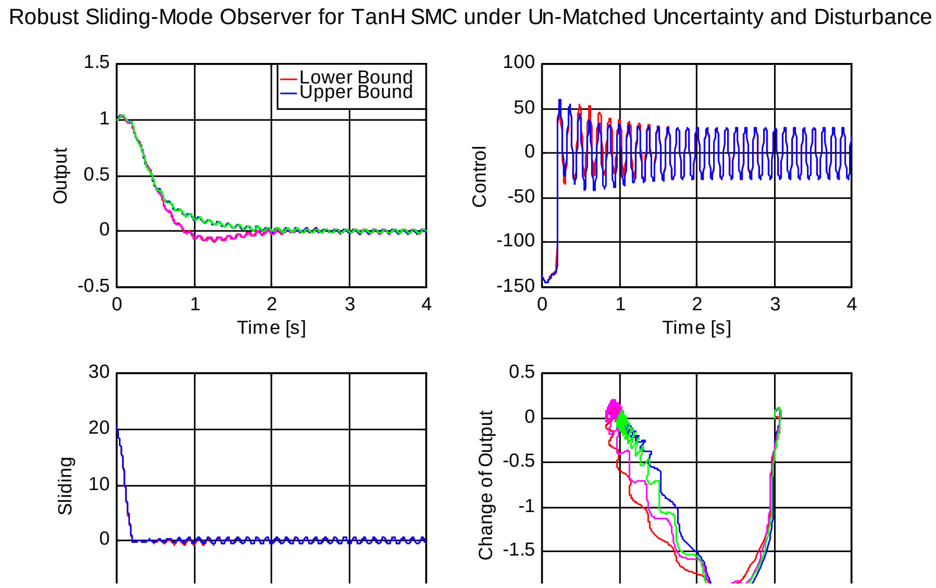


Fig. 4.20: Robust Sliding-Mode Observer for TanH SMC Controller in Example 4.7.

4.8.8. Example 4.8: Robust Sliding-Mode Observer under Converted Matched Uncertainties

Consider an un-matched uncertain linear system under external disturbance in Example 4.4 above, Section 4.7.2.2 converts the unmatched uncertain system into a matched system as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 40 & -3 & -8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.3 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 21 & 2 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$$

For the same maximum control effort, choose hyperplane eigenvalues as $\lambda_H = \begin{bmatrix} -6 & -6 \end{bmatrix}$, then

$$\mathbf{H} = \begin{bmatrix} 24.8276 & 8.2759 & 0.6897 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 34.8276 & 23.4483 & 2.7586 \end{bmatrix}$$

- Robust Linear SMC

For the same maximum control effort as in Example 4.4, choose $\delta = 6$, then Corollary 4.5 yields

$$\mathbf{K}_r = \begin{bmatrix} 148.9655 & 49.6552 & 4.1379 \end{bmatrix}$$

and choose $\delta_p = 0.2$ to have

$$\mathbf{K}_p = \begin{bmatrix} 76.1379 & 23.9310 & 2.0690 \end{bmatrix}$$

- Robust TanH SMC

Theorem 4.3 yields

$$\mathbf{K}_p = \begin{bmatrix} 21 & 2 & 0 \end{bmatrix}, \quad K_{p0} = 36$$

and choose $\delta = 2$, then

$$\mathbf{K}_r = \begin{bmatrix} 49.6552 & 16.5517 & 1.3793 \end{bmatrix}$$

Choose the observer dynamics 2 times faster than that of the robust linear SMC, then Theorem 4.11 yields

$$\mathbf{L} = \begin{bmatrix} 331.4542 \\ 840.4938 \\ 240.6432 \end{bmatrix}, \quad \mathbf{A}_e = \mathbf{A} - \mathbf{LC} = \begin{bmatrix} -331.4542 & 1 & 0 \\ -840.4938 & 0 & 1 \\ -200.6432 & -3 & -8 \end{bmatrix}$$

This observer can be used for both robust discontinuous and continuous sliding-mode controllers in the sequence.

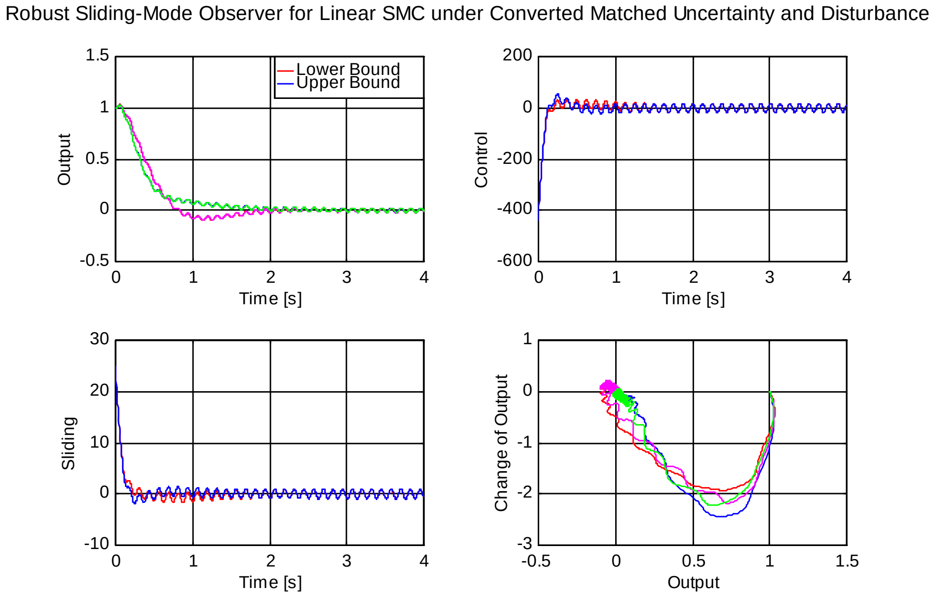


Fig. 4.21: Cf. Fig. 4.19, Robust Sliding-Mode Observer for Linear SMC Controller in Example 4.8.

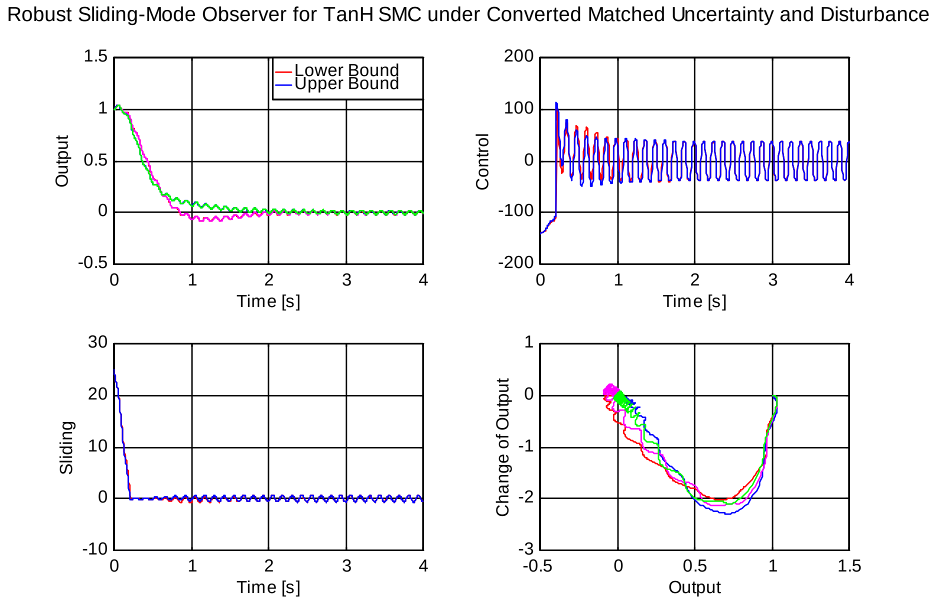


Fig. 4.22: Cf. Fig. 4.20, Robust Sliding-Mode Observer for TanH SMC Controller in Example 4.8.

Remark 4.15: Comparison between Unmatched and Converted Matched Uncertainty.

Performances of SMC's for converted matched uncertain system are similar to the original unmatched ones (Cf. Figs. 4.19 & 4.21, 4.20 & 4.22).

4.9. CONCLUSION

We have seen that the *sliding eigenvalues* are unchanged while the *system eigenvalues* are always changed under the perturbations. Even though the system eigenvalues vary with the perturbations, under *matching conditions*, the sliding eigenvalues are absolutely unchanged and equal to the hyperplane-eigenvalues chosen to be Hurwitz previously. Due to the stability criterion (Theorem 3.1), the system is stable regardless how large the perturbations are. Without the matching conditions, if the sliding-eigenvalues are still Hurwitz then the system is stable; but they may migrate into the right-half plane (RHP) and then the non-Hurwitz sliding-eigenvalues make the system unstable even though the sliding condition is still satisfied and the hyperplane is stable! In this case, we must do a stability test (Theorem 3.1).

We have identified 3 types of eigenvalues to clarify the mechanism of the sliding mode in a SMC with a sliding margin of δ: hyperplane-eigenvalues, sliding-eigenvalues and system-eigenvalues. The sliding-eigenvalues represent the sliding dynamics of the system state variables in the sliding mode. The hyperplane-eigenvalues are the desired sliding-eigenvalues. The first 2 types are applicable for both linear and nonlinear systems, the third is only for linear systems.

$$\begin{cases} \lambda_H = \text{eig}\left[\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA}\right] - \{0\} \\ \lambda_S = \text{eig}\left[\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\left(\mathbf{H}\tilde{\mathbf{B}}\right)^{-1}\mathbf{H}\tilde{\mathbf{A}}\right] - \{0\} \\ \lambda_C = \text{eig}[\mathbf{A} - \mathbf{BK}] = [-\delta, \quad \lambda_H] \quad \Rightarrow \quad \lambda_H = \text{eig}\,[\mathbf{A} - \mathbf{BK}] - \{-\delta\} \end{cases}$$

By the stability criterion (Theorem 3.1) with the design rule (Proposition 3.1), it is simple enough to do a stability test. If the sliding-eigenvalues are Hurwitz but its simulation is unstable, then we should reduce the sliding margin.

The unified design in Chapter 3 is extended to deal with perturbations in this chapter. Moreover, there is a unified approach to deal with discontinuous or continuous SMC, conventional or integral SMC. From a discontinuous control function of the discontinuous SMC, we can get a continuous pseudo-SMC by using either a saturated function or a TanH function in exact the same manner. For a linear SMC, it is only pseudo-SMC for the case of disturbances, but it is not for uncertainties. Also by the simulations, we have seen that the performance of the TanH SMC control is comparable to that of the linear SMC. Note that the former is a pseudo-sliding mode, the latter is not. Moreover, the latter is a linear control so the linear control theory can be applied for a more insight such as the closed-loop system eigenvalues.

We have also seen that it is possible for a linear control law for a certain class of nonlinear systems. For a more general class of nonlinear systems, a nonlinear system is linearized around an operating point, then the proposed robust linear SMC is applicable to that linearized system where the nonlinearity is considered as a special type of an uncertainty within the bound in that operating range.

By our new design of linear control function for linear systems under uncertainties, we can use the well-established linear control theory for assessments and for comparisons in order to have a closer look at the *true nature of SMC* as mentioned so far. In fact, the proposed robust linear SMC can be considered as a superset of the linear state-space control. Some other features are the followings:

(i) In the systems under no uncertainties, sliding-eigenvalues are equal to hyperplane-eigenvalues. Moreover, system-eigenvalues comprise hyperplane-eigenvalues and the negative of the *sliding margin*.

(ii) Invariance condition really means the sliding-eigenvalues are invariant to matched uncertainties and are equal to the hyperplane-eigenvalues, while the system-eigenvalues vary with the uncertainties. The Hurwitz sliding-eigenvalues do guarantee the system stability because they are equal to the Hurwitz hyperplane-eigenvalues. Moreover, by Theorem 4.5, the negative of the sliding margin and the hyperplane-eigenvalues are the system-eigenvalues at one boundary of matching uncertainties.

Chapter 4

Robust Sliding Mode Controller-Observer Design

Chapter 5

A New Robust Discrete-Time Sliding Mode Controller-Observer Design

5.1. INTRODUCTION

In the current discrete-time sliding-mode control (SMC), a convergence condition has been proposed in addition to the sliding condition in Sarpturk *et al* 1987 (Remark 5.1). There have been some comments on applicability of this approach in Kotta 1989, and these conditions have been shown to be sufficient but not necessary in Spurgeon 1990. In Utkin *et al* 1989, Furuta 1990, the proposed controls have been pseudo sliding mode with a boundary layer using discontinuous switch control function. In the current continuous-time SMC, a SMC can be obtained using a discontinuous switch control function and a pseudo-SMC is used to smooth out the discontinuity of the switch function in solution of the chattering problem. In Su *et al.* 1993, pre-filtering and post-filtering have been implemented to reduce chattering in sliding-mode controllers. In Spurgeon 1991 and Pieper *et al* 1992, robust discrete-time discontinuous pseudo SMC designs have been proposed for matched uncertain systems. In Pan *et al.* 1993, a discrete-time model has been transformed into a canonical one then a robust discrete-time SMC has been designed for this *transformed canonical model* under matched uncertainty. In Wang *et al.* 1994, a robust discrete-time SMC has been designed for a system where its *transformed model* under matched uncertainty.

In this chapter, a new robust discrete-time sliding mode controller (SMC) and observer design is presented for both discontinuous SMC (VSS) and continuous SMC control functions, it can be seen as a discrete-time version of the previous chapter. A high sampling rate is proved to be a necessary condition for a discrete-time SMC.

Based on the work in Pieper *et al* 1992 for matched uncertain systems, an alternative design will be presented without the constraint stated therein. On the basis of the new robust sliding mode controller design in the previous chapter, a new robust discrete-time sliding mode controller design will be proposed to deal with bot matched and unmatched uncertainty without chattering.

As a SMC is based on a state-space model, an I-action may be required to eliminate a steady-state error. We will also prove that the integral VSS control (discontinuous SMC) in Theorem 3.4 can be still applied to both robust discontinuous and linear continuous SMC controls in this chapter. Also as a state-space approach, SMC requires an observer to estimate unavailable states, we present a new robust discrete-time sliding-mode observer design.

As a state vector is necessary for SMC, an observer may be required for estimation of unavailable states. In Bondarev *et al.* 1985, a linear Luenberger observer has been used as an observer of a VSS controller for deterministic linear systems (no uncertainties). In Walcott *et al.* 1987 and Yaz *et al.* 1993, a Lyapunov sliding condition has been used to design an observer for a class of systems under matched uncertainty restricted to a certain system structure. In Slotine *et al.* 1987, a sliding patch condition has been used to have a region of direct attraction where uncertainty is not fully tackled. In fact, in Walcott *et al.* 1987, Slotine *et al.* 1987 and Yaz *et al.* 1993, to cope with uncertainty, a Lyapunov sliding condition has been employed to include a switching component into a linear Luenberger observer where a linearized model is used for a nonlinear system.

5.2. Digital Control Systems

Nowadays, most controllers are implemented as digital controllers rather than analog ones since digital controllers are much more flexible and they can be used to implement highly complicated control functions. Simple control functions can be implemented with analog controllers, however they are less reliable due to the drifting problem of op-amps, the leakage problem of capacitors and the aging problem of analog components such as resistors, capacitors etc.

The *z*-transform (Section 5.2.5) and a discretization of a state-space model (Section 5.2.7) will be used in the latter sections on discrete-time SMC. Although these results are well-known in the digital control literature, this section still exists due to its presentation on digital control systems in a compact, concise and logical manner. We first start with a reality of a sampling process to convert an analog signal into digital signal in deriving the *starred transform* which founds a basis to develop the *Z-transform*. The Z-transform will then define a mapping between the continuous-time *s*-plane and the discrete-time *z*-plane as a mathematical tool to design a digital controller, as the Laplace transform is used to design an analog controller.

5.2.1. Some Terminologies

In this subsection, we are concerned with the difference between *discrete-time* system and *digital* system while they both are a discretized system of an *analog* system. In the real world, all signals are *analog signals* whose amplitudes take *infinite* values within some ranges. Their values can be any real numbers with fraction and the difference of successive values is *infinitesimal*.

Analog signals are continuous-time signals. *Discrete-time signals*, or *discrete signals* for short, take finite values of any real numbers, hence the difference of successive values is finite and is a real number.

Digital signal is a subset of discrete-time signal. A values of a digital signal must be an *integer*, thus the difference of successive values is finite and is an integer. These integers are within the range determined by the length of registers in a ADC, the range is $\left[0 \sim 2^n\right]$or $\left[-2^{n-1} \sim 2^{n-1}-1\right]$ for a *n*-bit ADC.

Discrete-time control theory is used to design a discrete-time controller which is employed to implement a digital controller for a digital control system.

A continuous-time system is discretized using the discrete-time control theory, there is no digitization of a system since it is a special case of discretization as a real number is rounded off to an integer. An analog signal is digitized, also known as quantized, by an ADC, there is no discretization of an analog signal since there is no such converter available nowadays.

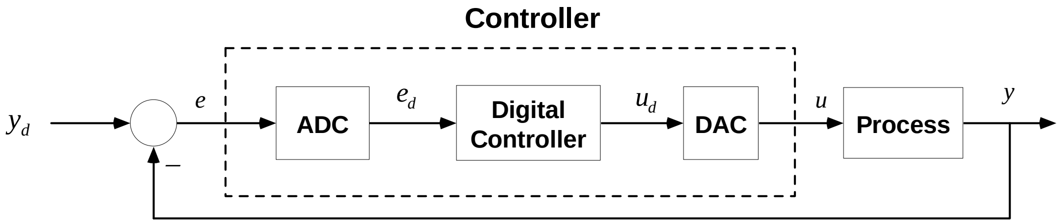


Fig. 5.1: Digital Control System

where y, y_d, e, and u are analog signals while e_d and u_d are digital signals. ADC and DAC *are analog-to-digital* and *digital-to-analog* converters, respectively. Digital controller can be either a micro-controller or a digital signal processor (DSP) where ADC and DAC are on-chip (internal) or a general purpose computer with external ADC/DAC. For 8-bit processors such as 8051 Intel family or 68HC11 Motorola family, ADC and DAC are also 8-bit. However, higher bit ADC's are expensive, 10-bit or 12-bit ADC are usually used for 16-bit processors such as 80196 Intel family, 68000 Motorola family, 32-bit 486-PC, 64-bit Pentium-PC.

5.2.2. ADC as Physical Sampler

Practically, ADC is a *zero-order-hold* sampler of *successive approximation* type

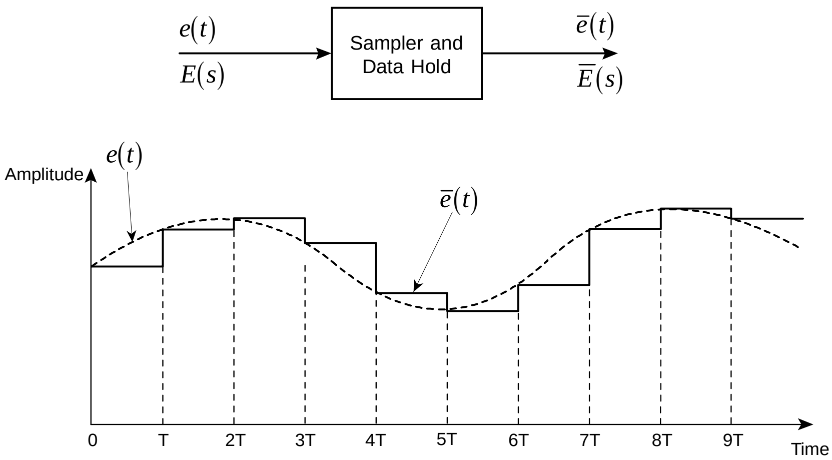


Fig. 5.2: Sample and Hold Signal

The *sampled and zero hold signal* $\bar{e}(t)$ can be mathematically expressed as

$$\bar{e}(t) = \sum_{k=-\infty}^{\infty} e(kT)\left\{\mathscr{U}(t-kT) - \mathscr{U}\left[t-(k+1)T\right]\right\} \tag{5.1}$$

where $\mathscr{U}(t)$ is the unit step function, that is

$$\mathscr{U}(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases} \tag{5.2}$$

Using the shifting theorem of Laplace transform gives

$$\overline{E}(s) = \sum_{k=-\infty}^{\infty} e(kT)\left\{\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right\} = \left(\frac{1-e^{-Ts}}{s}\right)\sum_{k=-\infty}^{\infty}\left[e(kT)e^{-kTs}\right] \tag{5.3}$$

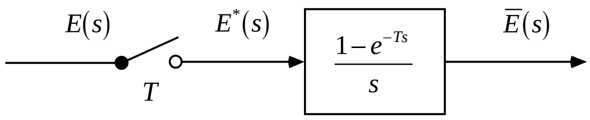
5.2.3. Starred Transform as Ideal Sampler

On the basis of Eq.(5.3), the *starred transform* is defined as

$$E^*(s) = \sum_{k=-\infty}^{\infty} e(kT)e^{-kTs} \tag{5.4}$$

then Eq.(5.3) can be written as

$$\overline{E}(s) = \left(\frac{1-e^{-Ts}}{s}\right) E^*(s) \tag{5.5}$$



Eq.(5.4) gives the inverse Laplace transform of $E^*(s)$ as

$$e^*(t) = \mathscr{L}^{-1}\left\{E^*(s)\right\} = \sum_{k=-\infty}^{\infty}\left[e(kT)\delta(t-kT)\right] \tag{5.6}$$

where $\delta(t)$ is a Dirac delta function.

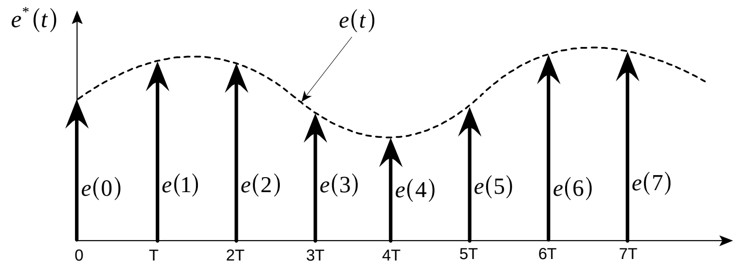


Fig. 5.3: A Representation of $e^*(t)$

5.2.4. Sampling Characteristics and Signal Reconstruction

This subsection founds a basis for choosing sampling rate and necessity of an analog low-pass filter as an anti-aliasing filter before ADC. Sampling characteristics and signal reconstruction is determined by the following theorem

Theorem 5.1: Laplace Transform of Sampled Signal

An alternative expression for the starred transform is given by

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s + jk\omega_s) \tag{5.7}$$

thus the Laplace transform of a sampled signal is equal to infinite sum of Laplace transform of the Laplace transform of its original signal. The s-plane may be divided into a *primary strip* and *complementary strips* as shown in Fig. 5.4.

Proof

Following the proof in Lockhart *et al.* 1989, let

$$\mathscr{S}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{5.8}$$

then Eq.(5.6) can be written as

$$e^*(t) = e(t).\mathscr{S}(t) \tag{5.9}$$

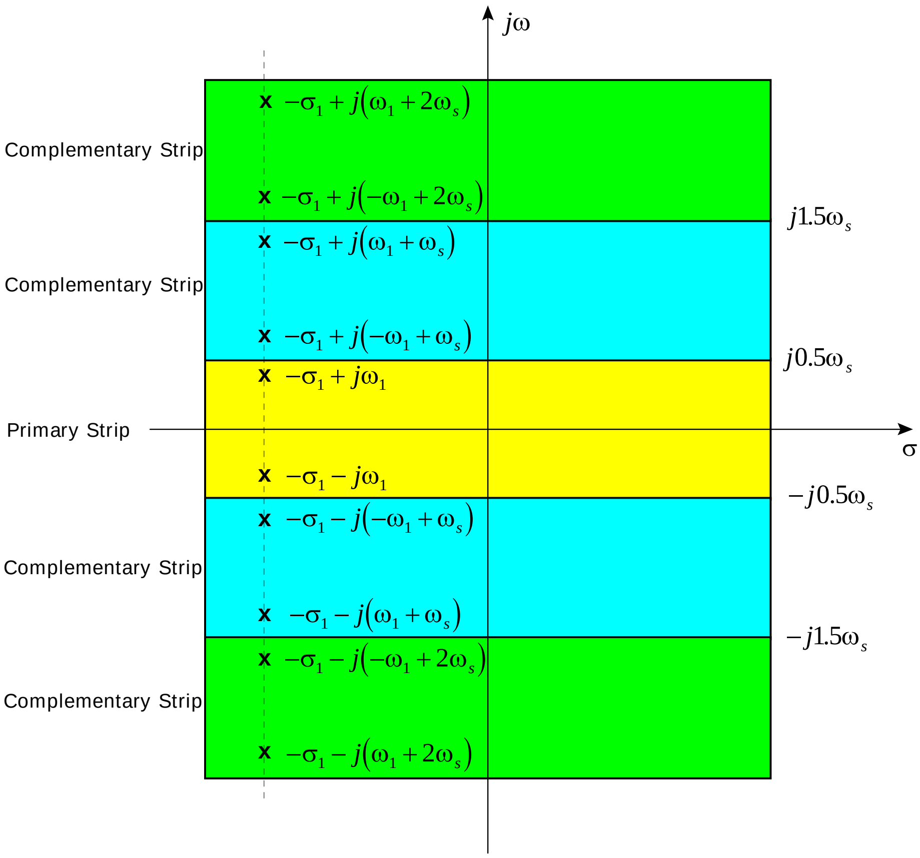


Fig. 5.4: Primary and Complementary Strips

Since $\mathscr{S}(t)$ is a periodic function $\delta(t)$ whose period is *T*, its Fourrier series is

$$\mathscr{S}(t) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\int_0^T \delta(t)e^{-j2\pi k\tau/T}d\tau\right)e^{j2\pi kt/T} = \frac{1}{T}\sum_{k=-\infty}^{\infty}e^{jk\omega_s t} \tag{5.10}$$

where

$$\omega_s = \frac{2\pi}{T} \tag{5.11}$$

thus Eq.(5.9) can be read as

$$e^*(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}e(t)e^{jk\omega_s t} \tag{5.12}$$

and its Laplace transform is

$$E^*(s) = \mathscr{L}\{e^*(t)\} = \int_0^{\infty}\frac{1}{T}\sum_{k=-\infty}^{\infty}e(t)e^{-(s-jk\omega_s)t}dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^{\infty}e(t)e^{-(s-jk\omega_s)t}dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}E(s+jk\omega_s)$$

Q.E.D.

5.2.4.1. Signal Reconstruction

To see the significance of Theorem 5.1, consider the case of frequency response, when $s = j\omega$ Eq.(5.7) gives

$$E^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}E\left[j(\omega + k\omega_s)\right] \tag{5.13}$$

thus the spectrum of sampled signal consists of an infinite number of replicas of the analog signal spectrum scaled by the factor $1/T$ and the frequency shifted by multiples of ω_s.

If the sampling time *T* is low enough such that $\omega_{\max} < \omega_s/2$, then the sampled spectrum retains the shape of the analog one and the analog signal can be reconstructed from its sampled spectrum within the band $[-\omega_s/2, \omega_s/2]$.

For signal reconstruction, the sampling rate must satisfy the sampling theorem, also known as Shannon Sampling Theorem, state that the sampling frequency ω_s must be al least equal twice the value of the highest significant frequency in the signal. Since an ideal low-pass reconstruction filter cannot be implemented, one rule of thumb is to choose *T* as one-tenth of the smallest process time constant or the desired closed-loop time constant. Another convenient rule suggests sampling at the rate of 6 to 10 times per cycle.

Proposition 5.1: Choice of Sampling Time
In this work, the sampling time is chosen such that $\omega_s \ge 10\omega_{\max}$. With this choice, ignoring the sampling effect will introduce a maximum tolerance of gain about 0.12% and of phase about 3^0.

Proof:

We have

$$e^{-sT} = \frac{e^{-sT/2}}{e^{sT/2}} = \frac{1 - \dfrac{sT}{2} + \dfrac{(sT)^2}{2^2 2!} - \cdots}{1 + \dfrac{sT}{2} + \dfrac{(sT)^2}{2^2 2!} + \cdots} \approx \frac{1 - \dfrac{sT}{2}}{1 + \dfrac{sT}{2}}$$

then Eq.(5.5) gives

$$G_h(s) = \frac{1-e^{-sT}}{s} \approx \frac{1}{s}\left(1 - \frac{1-\frac{sT}{2}}{1+\frac{sT}{2}}\right) = \frac{T}{\frac{T}{2}s+1}$$

Since the overall DC gain will be determined at the final design stage, the sampling time will be discarded to have

$$G_h(s) \approx \frac{1}{\frac{T}{2}s+1}$$

thus ignoring the sampling effect will introduce a maximum tolerance of

$$G_h(j\omega_{\max}) \approx \frac{1}{1+j\frac{0.1}{2}} = \frac{1}{1+j0.05} = 0.9988\angle -2.86^0$$

Q.E.D

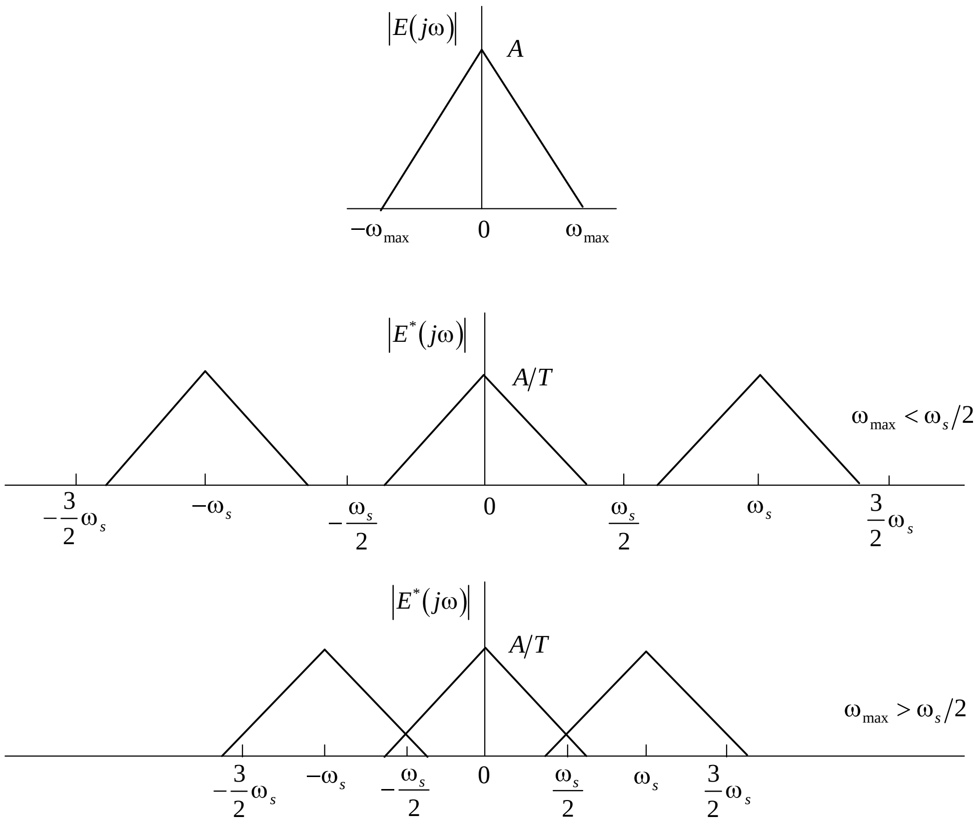


Fig. 5.5: Spectrum of Analog and Sampled Signal

5.2.4.2. Aliasing or Folding

Eq.(5.7) reveals that if $E(s)$ has a pole at p_0, then $E^*(s)$ will have poles at $(p_0 + jn\omega_s)$ where $n = 0, \pm 1, \pm 2, \cdots$ For example, suppose that $E(s)$ has one pair of poles located at $-a \pm j0.6\omega_s$, outside the primary strip. The sampling will then "fold" this pair back into the primary strip, to the location $-a \pm j0.4\omega_s$. So $e^*(t)$ contains components inside the primary strip that do not occur in $e(t)$. This property applies for poles only, not for zeros, since poles of a transfer function component in Eq.(5.7) will be poles of the whole transfer function; however, zeros of a component are not necessarily the zeros of the whole.

An anti-aliasing analog filter is used to solve this problem by filtering out all noise. It is a low-pass filter with the cut-off frequency is based on the highest frequency of the analog signal.

5.2.5. Z-Transform

On the basis of Eq.(5.4), the *Z transform* is defined as

$$E(z) = \mathcal{Z}\{e(k)\} = \sum_{k=-\infty}^{\infty} e(k) z^{-k} \tag{5.14}$$

where the mapping between the s-plane and the *z*-plane is defined as

$$z = e^{Ts} \quad \Leftrightarrow \quad s = \frac{1}{T}\ln(z) \tag{5.15}$$

since comparing Eqs.(5.4) and (5.14) yields

$$E^*(s) = E(z)\Big|_{z=e^{Ts}} \tag{5.16}$$

The Laplace transform is used in obtaining a *control transfer function* for implementing a analog controller. We will see that the Z-transform will be employed in obtaining a *control difference equation* for implementing a digital controller.

Two of the most important properties of the Z-transform are the linearity and the real are given below

$$\mathcal{Z}\{ae_1(k) + be_2(k)\} = a\mathcal{Z}\{e_1(k)\} + b\mathcal{Z}\{e_2(k)\} \tag{5.17}$$

and

$$\mathcal{Z}\{e(k-n)\} = z^{-n}\mathcal{Z}\{e(k)\} \tag{5.18}$$

where n is a positive integer.

Since by the definition of the Z-transform, we have

$$\mathcal{Z}\{ae_1(k) + be_2(k)\} = \sum_{k=-\infty}^{\infty}[ae_1(k) + be_2(k)]z^{-k} = a\sum_{k=-\infty}^{\infty} e_1(k)z^{-k} + b\sum_{k=-\infty}^{\infty} e_2(k)z^{-k} = a\mathcal{Z}\{e_1(t)\} + b\mathcal{Z}\{e_2(t)\}$$

and

$$\mathcal{Z}\{e(k-n)\} = \sum_{k=-\infty}^{\infty}[e(k-n)]z^{-k} = \sum_{m=-\infty}^{\infty} e(m)z^{-(n+m)} = z^{-n}\sum_{k=-\infty}^{\infty} e(m)z^{-m} = z^{-n}\mathcal{Z}\{e(k)\}$$

where $m = k - n$.

5.2.6. Some Typical S-Z Transformations

For completeness, this subsection presents a brief derivation of 2 well-known transformations: backward difference method and bilinear method. The Z-transform defines the mapping in Eq.(5.15) as the exact transformation between the *s*-plane and the *z*-plane. This transform should be used to convert a continuous-time system into a discrete-time one. A direct substituting Eq.(5.15) into a transfer function in *s*-plane will produce an infinite difference equation due to the Taylor expansion of the natural logarithmic function. This necessitates an approximate transformation to obtain finite difference equations if the substitution method is used *in place* of the z-transform. From a single term s, we can have s and $1/s$ corresponding to a differentiator and an integrator, respectively.

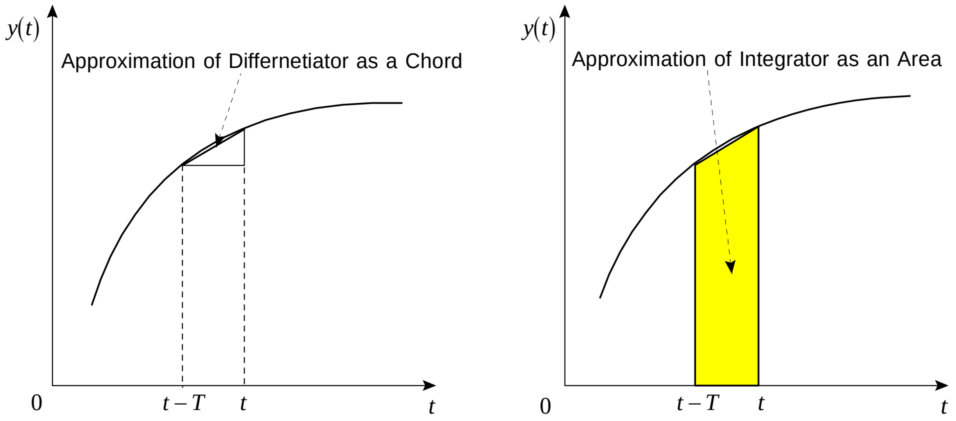


Fig. 5.6: Approximate Transformations for Differentiator and Integrator

The backward difference transformation approximates a differentiator as the slope of a chord

$$\frac{d}{dt}y(t) \approx \frac{y(t)-y(t-T)}{T}$$

taking Laplace transform gives

$$sY(s) \approx \frac{Y(s)-e^{-Ts}Y(s)}{T}$$

so

$$s \approx \frac{1-e^{-Ts}}{T} = \frac{1-z^{-1}}{T} \quad \Leftrightarrow \quad z \approx \frac{1}{1-Ts} \qquad (5.19)$$

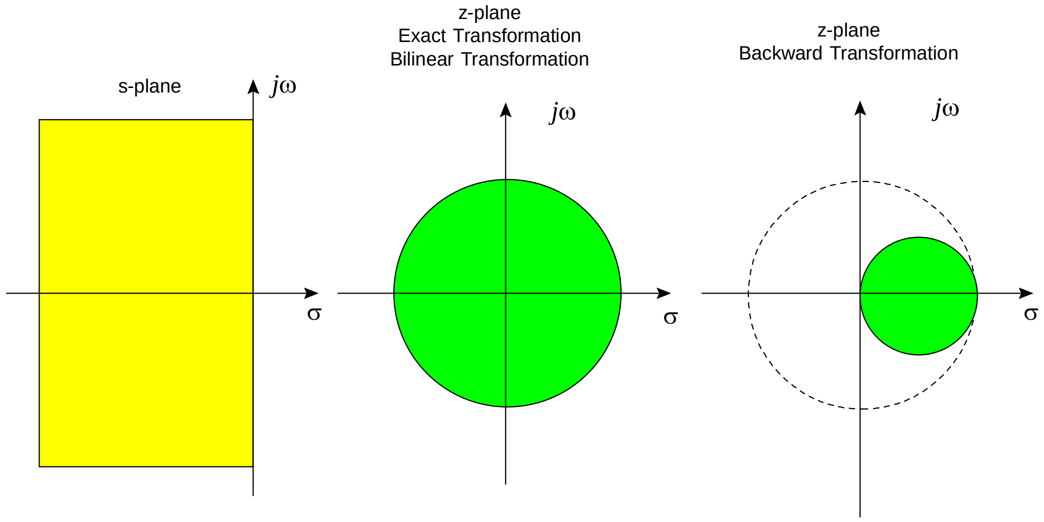


Fig. 5.7: Stability Regions

The bilinear transformation approximates an integrator as the area of a trapezoidal

$$\int_0^t y(\tau).d\tau - \int_0^{t-T} y(\tau).d\tau \approx \frac{T}{2}\left[y(t-T)+y(t)\right]$$

taking Laplace transform gives

$$\frac{1}{s}Y(s) - \frac{1}{s}e^{-Ts}Y(s) \approx \frac{T}{2}\left[e^{-Ts}Y(s) + Y(s)\right]$$

thus

$$\frac{1}{s} \approx \frac{T}{2}\frac{1+e^{-Ts}}{1-e^{-Ts}} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \quad \Leftrightarrow \quad z \approx \frac{(T/2)+s}{(T/2)-s} \tag{5.20}$$

The stability region in the *s*-plane is the RHP, to find its mapping in the *z*-plane, substituting the frequency contour $s = j\omega$ into Eqs.(5.15), (5.20) and (5.21) produces

$$z = e^{j\omega T} \tag{5.21}$$

$$z = \frac{1}{1-j\omega T} = \frac{1}{2} + \frac{1}{2}\frac{1+j\omega T}{1-j\omega T} = \frac{1}{2} + \frac{1}{2}e^{j\theta}, \quad \theta = 2\tan^{-1}(2\omega T) \tag{5.22}$$

$$z \approx \frac{(T/2)+j\omega T}{(T/2)-j\omega T} = e^{j\theta}, \quad \theta = 2\tan^{-1}(2\omega) \tag{5.23}$$

Thus a stable controller could be unstable under the backward difference transformation. This is a limitation of this transformation.

5.2.7. Discretization of Continuous-Time State-Space Equations

The *z*-transform is used to discretize a continuous-time transfer-function model for design a discrete-time controller in the *z*-domain. Alternatively, a transfer-function model can be modified by taking into account the sampling process, and a continuous-time controller can be designed in the *s*-plane then discretized to obtain a discrete-time controller. To discretize a continuous-time state-space model, from the *digital control literature*, we have the following theorem:

Theorem 5.2: Discretization of State-Space Model

Consider the continuous-time state equation and output equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \tag{5.24}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{5.25}$$

under the assumption of zero-order hold that

$$\mathbf{u}(t) = \mathbf{u}(kT), \qquad kT \le t \le (k+1)T \tag{5.26}$$

then a discrete-time representation of Eq.(5.24) will take the form

$$\mathbf{x}\left[(k+1)T\right] = \mathbf{A}_d(T).\mathbf{x}(kT) + \mathbf{B}_d(T)\mathbf{u}(kT) \tag{5.27}$$

$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT) \tag{5.28}$$

where

$$\mathbf{A}_d(T) = e^{\mathbf{A}_c T} \tag{5.29}$$

and

$$\mathbf{B}_d(T) = \left(\int_0^T e^{\mathbf{A}_c t}\,dt\right)\mathbf{B}_c \tag{5.30}$$

with T is a constant of sampling time, $\mathbf{A}_d(T)$ *and* $\mathbf{B}_d(T)$ are thus constant matrices.

Proof

Following the proof in Ogata 1987, the Laplace transform of Eq.(5.24) give

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}_c\mathbf{X}(s) + \mathbf{B}_c\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A}_c)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}_c\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c\mathbf{U}(s) \tag{5.31}$$

Note that

$$(s\mathbf{I} - \mathbf{A}_c)^{-1} = \sum_{n=0}^{\infty}\frac{\mathbf{A}_c^n}{s^{n+1}}, \qquad \mathbf{A}_c^0 = \mathbf{I} \tag{5.32}$$

hence its inverse Laplace transform is

$$\mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A}_c)^{-1}\right\} = \sum_{n=0}^{\infty}\frac{(\mathbf{A}_c t)^n}{n!} = e^{\mathbf{A}_c t}, \qquad 0! = 1 \tag{5.33}$$

thus Eq.(5.32) becomes

$$\mathbf{X}(s) = e^{\mathbf{A}_c t}\mathbf{x}(0) + e^{\mathbf{A}_c t}\mathbf{B}_c\mathbf{U}(s) \tag{5.34}$$

then its inverse Laplace transform is

$$\mathbf{x}(t) = e^{\mathbf{A}_c t}\mathbf{x}(0) + e^{\mathbf{A}_c t}\int_0^t e^{-\mathbf{A}_c \tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{5.35}$$

This equation gives

$$\mathbf{x}[(k+1)T] = e^{\mathbf{A}_c(k+1)T}\mathbf{x}(0) + e^{\mathbf{A}_c(k+1)T}\int_0^{(k+1)T} e^{-\mathbf{A}_c \tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{5.36}$$

and

$$\mathbf{x}(kT) = e^{\mathbf{A}_c kT}\mathbf{x}(0) + e^{\mathbf{A}_c kT}\int_0^{kT} e^{-\mathbf{A}_c \tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{5.37}$$

Multiplying Eq.(5.37) by $e^{\mathbf{A}_c T}$ and subtracting from Eqs.(5.26) and (5.36) yields

$$\mathbf{x}[(k+1)T] = e^{\mathbf{A}_c T}\mathbf{x}(kT) + e^{\mathbf{A}_c(k+1)T}\int_{kT}^{(k+1)T} e^{-\mathbf{A}_c \tau}\mathbf{B}_c\mathbf{u}(kT)d\tau$$

then

$$\mathbf{x}[(k+1)T] == e^{\mathbf{A}_c T}\mathbf{x}(kT) + \left(\int_0^T e^{\mathbf{A}_c t}\mathbf{B}_c dt\right)\mathbf{u}(kT), \qquad t = (k+1)T - \tau \tag{5.38}$$

Substituting $t = kT$ in Eq.(5.25) gives Eq.(5.28).

Q.E.D.

5.3. DISCRETE-TIME SLIDING HYPERPLANE DESIGN

The sliding-mode control has 2 phases of dynamics: the reaching and the sliding modes. The sliding condition forces the system states to reach the hyperplane and keeps them sliding on it. In this section, the hyperplane design and the sliding dynamics will be presented.

Proposition 5.2: Discrete-Time Sliding-Mode Controller Design

If design parameters such as hyperplane, control function (continuous and discontinuous SMC) are available in continuous-time domain, then they can be used for the discretized system to keep the discrete-time controller parameters as close as the continuous-time counterparts. Otherwise, if they are unavailable in case of continuous-time model is unavailable, then the rest of this chapter will be used to design a discrete-time sliding-mode controller.

A *discrete-tim*e sliding-mode observer must be designed since it will be used in a discrete-time observer equation to estimate system states.

Proof:

By Theorem 5.2, we have

$$\begin{cases} \mathbf{x}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

then

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}$$

where **A, B** are discretizations of $\mathbf{A}_c, \mathbf{B}_c$, respectively.

In this sense, consider

$$\begin{cases} \mathbf{x}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \\ \mathbf{s}(t) = \mathbf{H}\mathbf{x}(t) \\ \mathbf{u}(t) = \begin{cases} -\mathbf{K}\mathbf{x}(t) \\ -\mathbf{K}_e\mathbf{x}(t) - \mathbf{K}_r\left|\mathbf{x}(t)\right|.\operatorname{sign}\left[\mathbf{s}(t)\right] \end{cases} \end{cases}$$

then we have

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{s}(k) = \mathbf{H}\mathbf{x}(k) \\ \mathbf{u}(k) = \begin{cases} -\mathbf{K}\mathbf{x}(k) \\ -\mathbf{K}_e\mathbf{x}(k) - \mathbf{K}_r\left|\mathbf{x}(k)\right|.\operatorname{sign}\left[\mathbf{s}(k)\right] \end{cases} \end{cases}$$

Q.E.D.

5.3.1. Hyperplane Equation

The exact transformation in Eq.(5.15) converts a desired continuous-time hyperplane-eigenvalues $\mathbf{P}_c$ into a discrete-time counterpart $\mathbf{P}$, then Chapter 2 will be used with the discrete-time model $\{\mathbf{A}, \mathbf{B}\}$ to design a hyperplane for a discrete-time sliding-mode controller.

Continuous-Time Hyperplane Design	Discrete-Time Hyperplane Design
$\mathbf{H}_c = hyper(\mathbf{A}_c, \mathbf{B}_c, \mathbf{P}_c)$	$\mathbf{H} = hyper(\mathbf{A}, \mathbf{B}, \mathbf{P})$

where

$\mathbf{P} = e^{T\mathbf{P}_c}$;

hyper is exactly the same hyperplane design procedure presented in Chapter 2 (Remark 2.8).

5.3.2. Equivalent Control

Consider a hyperplane

$$s(k) = \mathbf{H}\mathbf{x}(k)$$

then

$$\Delta s(k) = s(k+1) - s(k) = \mathbf{H}.\left[\mathbf{x}(k+1) - \mathbf{x}(k)\right] = \mathbf{H}(\mathbf{A} - \mathbf{I}).\mathbf{x}(k) + \mathbf{HB}.u(k)$$

where

$$\mathbf{H} \in \Re^{1\times n}, \qquad \mathbf{x} \in \Re^{n\times 1}$$

Assume $(\mathbf{HB})$ is invertible, then the equivalent control is defined as

$$u_{eq}(k) = -(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}).\mathbf{x}(k) \tag{5.39}$$

thus

$$\Delta s(k) = s(k+1) - s(k) = \mathbf{HB}.\left[u(k) - u_{eq}(k)\right] \tag{5.40}$$

It is important to note that in the sliding mode

$$s(k) = 0 \Rightarrow \Delta s(k) = 0 \Rightarrow u(k) = u_{eq}(k).$$

5.3.3. Sliding-Eigenvalues

We have the following corollary as a discretization of sliding-eigenvalues in Theorem 2.2

Theorem 5.3: Discrete-Time Sliding Eigenvalues

Consider a SISO discrete-time state equation

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,u(k)$$

and a hyperplane

$$s(k) = \mathbf{H}\,\mathbf{x}(k)$$

In the sliding mode

$$u(k) = u_e(k) = -\mathbf{K}_e.\mathbf{x}(k), \qquad \mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I})$$

then the sliding-eigenvalue λ_S can be determined by

$$\text{eig}\left[\mathbf{A} - \mathbf{BK}_e\right] = \text{eig}\left[\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I})\right] = \{1, \quad \lambda_S\} \tag{5.41}$$

Proof:

For a reduced order of an *n*-ordered system, there are only $(n-1)$ sliding-eigenvalues λ_S, so we need to prove that the remaining *n*-th eigenvalue of Eq.(5.41) is *zero*. By definition of eigenvalues: $\mathrm{eig}(\mathbf{M}) = \{\lambda_i | \det[\lambda_i \mathbf{I} - \mathbf{M}] = 0\}$, thus we have to prove that $\det\{\mathbf{I} - [\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A}-\mathbf{I})]\} = 0$ or equivalently

$$\det\left[\tilde{\mathbf{A}} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\tilde{\mathbf{A}}\right] = 0 .$$

where

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{I}$$

Since eigenvalues are invariant under a *similarity transformation*, we will use the similarity transformation as in Utkin *et al.* 1979.

Consider a system after the similarity transformation

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \mathbf{B}.u(k)$$

and a hyperplane

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

where

$$\mathbf{x},\mathbf{B} \in \Re^{n\times 1}, \quad \tilde{\mathbf{A}} \in \Re^{n\times n}, \quad u,s \in \Re, \quad \underline{\underline{\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}}}$$

and

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \quad \tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix}, \quad \tilde{\mathbf{A}}_{11} \in \Re^{(n-1)\times(n-1)}, \quad \tilde{\mathbf{A}}_{12} \in \Re^{(n-1)\times 1}, \quad \tilde{\mathbf{A}}_{21}, \mathbf{x}_1 \in \Re^{1\times(n-1)}, \quad \tilde{\mathbf{A}}_{22}, \mathbf{x}_2 \in \Re$$

We have

$$\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}\left(\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} = \mathbf{H}_2^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{H}_1\tilde{\mathbf{A}}_{11} + \mathbf{H}_2\tilde{\mathbf{A}}_{21}, & \mathbf{H}_1\tilde{\mathbf{A}}_{12} + \mathbf{H}_2\tilde{\mathbf{A}}_{22} \end{bmatrix}$$

thus

$$\left[\tilde{\mathbf{A}} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\tilde{\mathbf{A}}\right] = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} - \mathbf{H}_2^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{H}_1\tilde{\mathbf{A}}_{11} + \mathbf{H}_2\tilde{\mathbf{A}}_{21}, & \mathbf{H}_1\tilde{\mathbf{A}}_{12} + \mathbf{H}_2\tilde{\mathbf{A}}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ -\mathbf{H}_2^{-1}\mathbf{H}_1\tilde{\mathbf{A}}_{11}, & -\mathbf{H}_2^{-1}\mathbf{H}_1\tilde{\mathbf{A}}_{12} \end{bmatrix}$$

By the property of a determinant

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \begin{cases} |\mathbf{A}|.|\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}|, & \text{if } |\mathbf{A}| \neq 0 \\ |\mathbf{D}|.|\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}|, & \text{if } |\mathbf{D}| \neq 0 \end{cases} \quad \Rightarrow \quad \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{vmatrix} = |\mathbf{A}|.|\mathbf{D}|$$

then

$$\det\left[\tilde{\mathbf{A}} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}\tilde{\mathbf{A}}\right] = \det\begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ -\mathbf{H}_2^{-1}\mathbf{H}_1\tilde{\mathbf{A}}_{11}, & -\mathbf{H}_2^{-1}\mathbf{H}_1\tilde{\mathbf{A}}_{12} \end{bmatrix} = 0$$

Therefore

$$\left|\lambda\mathbf{I} - \left[\mathbf{A} - \mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A}-\mathbf{I})\right]\right| = (\lambda - 1)\times\mathscr{P}\left(\lambda^{n-1}\right) \Rightarrow \lambda_H = \left\{\lambda_i | \mathscr{P}\left(\lambda^{n-1}\right) = 0\right\} \tag{5.42}$$

Q.E.D.

5.4. A Robust Discrete-Time Discontinuous SMC (VSS) Design

Based on the work in Pieper *et al* 1992 for matched uncertain systems, we propose the following theorem without the constraint stated therein

Theorem 5.4: Robust Discrete-Time Discontinuous SMC Design under Matched Uncertainty

Consider a discrete-time linear system under matched parametric uncertainty (Corollary 4.1)

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\{u(k) + v(k)\} \tag{5.43}$$

where the disturbance $v(k)$ must satisfy the following bound

$$\max_i |v_i| \le \bar{v} \tag{5.44}$$

and a hyperplane equation

$$s(k) = \mathbf{H}\,\mathbf{x}(k)$$

where

$$\mathbf{x}, \mathbf{B} \in \Re^{n\times 1},\ \mathbf{A} \in \Re^{n\times n},\ \mathbf{H} \in \Re^{1\times n},\ u, s \in \Re$$

To satisfy the discrete-time sliding condition

$$|s(k+1)| < |s(k)| \ \Rightarrow\ s^2(k+1) < s^2(k) \tag{5.45}$$

a control can be determined by

$$\underline{\underline{u(k) = u_e(k) + u_{rp}(k)}} \tag{5.46}$$

where

- equivalent control

$$u_e(k) = -\mathbf{K}_e\,\mathbf{x}(k), \quad \mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}) \tag{5.46.a}$$

- reaching and perturbation control

$$u_{rp}(k) = -\mathbf{K}_{rp}\,\mathbf{x}(k), \quad k_{rpi} = \begin{cases} \delta\,\mathrm{sgn}\{\mathbf{HB}s(k)x_i\}, & \text{if } |s(k)| > \beta, \quad \beta = \tfrac{1}{2}|\mathbf{HB}|\left(\bar{v} + \delta\sum_{i=1}^{n}|x_i|\right) \\ 0, & \text{else} \end{cases} \tag{5.46.b}$$

with the design parameter $\delta > 0$ can be considered as a sliding margin and determined due to a reasonable control effort as in the continuous domain.

Proof:

Let

$$s = s(k), \quad \mathbf{x} = \mathbf{x}(k), \quad u_q = \mathbf{HB}\left[u_{rp}(k) + v(k)\right]$$

then from Eqs.(5.40) and (5.46), we have

$$s(k+1) - s(k) = \mathbf{HB}\left[u_{rp}(k) + v(k)\right] = u_q$$

so

$$s(k+1) + s(k) = u_q + 2s$$

thus

$$s^2(k+1) - s^2(k) = u_q\left(u_q + 2s\right)$$

To satisfy the sliding condition Eq.(5.45), we may choose

$$|u_q| < 2|s| \ \Rightarrow\ \left|\mathbf{HB}\left(u_{rp} + v\right)\right| < 2|s| \ \Rightarrow\ |s| > \tfrac{1}{2}|\mathbf{HB}|\left|u_{rp} + v\right|$$

By Eq.(5.46.b), we obtain

$$|s| > \tfrac{1}{2}|\mathbf{HB}|\left|\bar{v} + \delta\sum_{i=1}^{n}|x_i|\right|$$

If

$$|s| \le \tfrac{1}{2}|\mathbf{HB}|\left|\bar{v} + \delta\sum_{i=1}^{n}|x_i|\right|$$

then

$$|\mathbf{HB}\, v| \le |\mathbf{HB}\, \bar{v}| \le |\mathbf{HB}|\left|\bar{v} + \delta\sum_{i=1}^{n}|x_i|\right| = 2\,|s|$$

or

$$|\mathbf{HB}\, v| \le 2\,|s|$$

by Eq.(5.46.b), we have

$$\mathbf{K}_r = \mathbf{0}$$

so

$$s^2(k+1) - s^2(k) = \mathbf{HB}\, v\left(\mathbf{HB}\, v + 2s\right)$$

hence

$$s^2(k+1) \le s^2(k)$$

Q.E.D.

Remark 5.1: Validity of Convergence Condition in Discrete-Time SMC

In Sarpturk *et al* 1987, both sliding and convergence conditions have been proposed as

$$\begin{cases} s(k)\,\{s(k+1) - s(k)\} < 0: \text{ sliding condition} \\ s(k)\,\{s(k+1) + s(k)\} \ge 0: \text{ convergence condition} \end{cases} \tag{5.47}$$

The sliding condition in Eq.(5.45) can be read

$$[s(k+1) - s(k)][s(k+1) + s(k)] < 0 \;\Rightarrow\; s^2(k)\,[s(k+1) - s(k)][s(k+1) + s(k)] < 0$$

or

$$\{s(k)\,[s(k+1) - s(k)]\}\,\{s(k)\,[s(k+1) + s(k)]\} < 0$$

so

$$\begin{cases} s(k)\,[s(k+1) - s(k)] < 0 \\ s(k)\,[s(k+1) + s(k)] > 0 \end{cases} \quad \textbf{or} \quad \begin{cases} s(k)\,[s(k+1) - s(k)] > 0 \\ s(k)\,[s(k+1) + s(k)] < 0 \end{cases} \tag{5.48}$$

Thus Eq.(5.47) is only part of the discrete-time sliding condition Eq.(5.45) so it may be too conservative to be used.

5.5. A NEW ROBUST DISCRETE-TIME LINEAR SLIDING MODE CONTROLLER DESIGN

In this section, a robust SMC design will be proposed for uncertain linear systems under parametric uncertainty and external disturbance.

5.5.1. Robust Discrete-Time Linear SMC under External Disturbance

To design a SMC under external disturbance, we have the following theorem

Theorem 5.5: Robust Discrete-Time Linear SMC Design under External Disturbance

Consider a linear system under disturbances

$$\mathbf{x}(k+1) = \mathbf{A}.\mathbf{x}(k) + \mathbf{B}.u(k) + \mathbf{W}.v(k), \qquad |v| \le \bar{v} \tag{5.49}$$

with a hyperplane equation

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

where

$$\mathbf{x}, \mathbf{B}, \mathbf{W} \in \Re^{n\times 1}; \quad \mathbf{A} \in \Re^{n\times n}; \quad \mathbf{H} \in \Re^{1\times n}; \quad u, v, s \in \Re$$

then a robust linear SMC function can be found from

$$\underline{\underline{u = -\mathbf{K}.\mathbf{x}, \quad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p}} \tag{5.50}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}) \tag{5.50.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\rho.\mathbf{H} \tag{5.50.b}$$

- perturbation control

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\rho_p \sup|\mathbf{HW}|.\bar{v}.\mathbf{H} \tag{5.50.c}$$

if the following sampling condition is satisfied (Remark 5.2)

$$0 < \rho + \rho_p \sup|\mathbf{HW}|.\bar{v} < 1 \tag{5.50.d}$$

and if the following disturbance condition is satisfied

$$|s(k)| > \sup\left(\frac{\rho_p s \sup|\mathbf{HW}|.\bar{v}}{\rho + \rho_p \sup|\mathbf{HW}|.\bar{v}}, \; \frac{\rho_p \sup|\mathbf{HW}|.\bar{v}}{1 - \left[\rho + \rho_p \sup|\mathbf{HW}|.\bar{v}\right]} \right) \tag{5.51}$$

where ρ, ρ_p are discrete-time sliding margins equivalent to the continuous-time sliding margins δ, δ_p.

Proof

From Eqs.(5.49) and (5.50), we have

$$s(k+1) - s(k) = \mathbf{H}.\left[\mathbf{x}(k+1) - \mathbf{x}(k)\right] = \mathbf{H}.(\mathbf{A} - \mathbf{I}).\mathbf{x}(k) + \mathbf{HB}.u(k) + \mathbf{HW}.v(k)$$

or

$$\begin{cases} s(k+1) - s(k) = -\bar{\alpha}.s(k) + \xi(k) \\ s(k+1) + s(k) = (1 - \bar{\alpha}).s + \xi(k) \end{cases}$$

so

$$s^2(k+1) - s^2(k) = -\bar{\alpha}(1 - \bar{\alpha}).s^2(k) + (1 - 2\bar{\alpha}).\xi(k).s(k) + \xi^2(k) \tag{5.52}$$

where

$$\bar{\alpha} = \rho + \rho_p \sup|\mathbf{HW}|.\bar{v}, \qquad \xi(k) = \mathbf{HW}.v(k) \tag{5.53}$$

By the sampling condition in Eq.(5.50.d), we have

$$0 < \overline{\alpha} < 1 \quad \Rightarrow \quad \overline{\alpha}(1-\overline{\alpha}) > 0$$

then we consider the following 2 cases

- If $\dfrac{-\xi(k)}{1-\overline{\alpha}} < \dfrac{\xi(k)}{\overline{\alpha}}$, then

$s(k)$		$\dfrac{-\xi(k)}{1-\overline{\alpha}}$		$\dfrac{\xi(k)}{\overline{\alpha}}$	
$s^2(k+1)-s^2(k)$	$-$	0	+	0	$-$

- If $\dfrac{\xi(k)}{\overline{\alpha}} < \dfrac{-\xi(k)}{1-\overline{\alpha}}$, then

$s(k)$		$\dfrac{\xi(k)}{\overline{\alpha}}$		$\dfrac{-\xi(k)}{1-\overline{\alpha}}$	
$s^2(k+1)-s^2(k)$	$-$	0	+	0	$-$

So, in the worst case, the discrete-time sliding condition is satisfied if

$$|s(k)| > \sup\left(\frac{|\xi|}{\overline{\alpha}}, \ \frac{|\xi|}{1-\overline{\alpha}}\right)$$

By Eq.(5.53), we obtain Eq.(5.51).

Q.E.D.

5.5.2. Robust Discrete-Time Linear SMC under Parametric Uncertainties

In this subsection, a general case of parametric uncertainty is considered first, the cases under no uncertainty and under matched uncertainty are the special cases.

5.5.2.1. Matched and Unmatched Parametric Uncertainties

We propose the following theorem for *both non-matched and matched* uncertain dynamical systems.

Assumption 5.1: System Constraint on Parametric Variation

A system matrix $\tilde{\mathbf{B}}$ takes any variation such that the polarity of $(\mathbf{H}\tilde{\mathbf{B}})$ is unchanged, *ie.*

$$\operatorname{sgn}(\mathbf{H}\tilde{\mathbf{B}}) = \operatorname{sgn}(\mathbf{H}\mathbf{B}) \quad \Rightarrow \quad |\Delta\tilde{\mathbf{B}}| < \Delta\mathbf{B} \tag{5.54}$$

where

$$\tilde{\mathbf{B}} = \mathbf{B} + \Delta\tilde{\mathbf{B}}$$

Under Assumption 5.1, we propose the following theorem to design a robust linear discrete-time sliding mode controller for both matched and unmatched uncertainties.

Theorem 5.6: Robust Discrete-Time Linear SMC Design under Parametric Uncertainty

Consider an uncertain dynamical linear discrete-time system

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k) \tag{5.55}$$

with a hyperplane

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B}$$

with

$$\mathbf{x} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B}, \Delta\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re$$

then, under Assumption 5.1, a robust discrete-time linear sliding mode control function can be determined by

$$u(k) = -\mathbf{K}.\mathbf{x}(k), \quad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p \tag{5.56}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}) \tag{5.56.a}$$

- reaching control

$$\mathbf{K}_r = \left(\mathbf{H}\overline{\mathbf{B}}\right)^{-1}\rho.\mathbf{H} \tag{5.56.b}$$

- perturbation control

$$\mathbf{K}_p = \left[\frac{\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}\left|\mathbf{K}_e + \mathbf{K}_r\right|\right)}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\right] \otimes \operatorname{sgn}(\mathbf{HB}) \tag{5.56.c}$$

if the following sampling condition is satisfied (Remark 5.2)

$$(1-\rho) \ge \left(1 + \frac{\sup|\mathbf{HB}|}{\inf|\mathbf{HB}|}\right) \max_i \frac{\Delta_i}{2.|h_i|} \tag{5.57}$$

where

$$\Delta_i = |\mathbf{H}|.\left(\Delta\mathbf{A}_{i(col)} + \Delta\mathbf{B}.\left|K_{ei} + K_{ri}\right|\right) \tag{5.57.a}$$

$$\mathbf{K}_e, \mathbf{K}_r, \mathbf{K}_p \in \Re^{1\times n}, \quad |\mathbf{M}| = \left[\left|m_{ij}\right|\right], \quad \left[p_{ij}\right] \otimes \left[q_{ij}\right] = \left[p_{ij}.q_{ij}\right]$$

Proof

To keep equivalent and reaching controls similar to the case without uncertainty, we will prove a perturbation control in Eq.(5.57.c) based on Eqs.(5.57.a) and (5.57.b).

Let

$$\mathbf{K}_{er} = \mathbf{K}_e + \mathbf{K}_r$$

by Eq.(5.57), we have

$$s(k+1) - s(k) = \mathbf{H}.\left[\mathbf{x}(k+1) - \mathbf{x}(k)\right] = \mathbf{H}.\left[\left(\tilde{\mathbf{A}} - \mathbf{I}\right).\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k)\right] = \mathbf{H}\left(\tilde{\mathbf{A}} - \mathbf{I}\right).\mathbf{x}(k) + \mathbf{H}\tilde{\mathbf{B}}.u(k)$$

$$s(k+1) - s(k) = \mathbf{H}(\mathbf{A} - \mathbf{I}).\mathbf{x}(k) - \mathbf{HB}.\left(\mathbf{K}_{er} + \mathbf{K}_p\right).\mathbf{x}(k) = \mathbf{H}\left(\tilde{\mathbf{A}} - \mathbf{I}\right).\mathbf{x}(k) - \mathbf{H}\tilde{\mathbf{B}}.\mathbf{K}_{er}.\mathbf{x}(k) - \mathbf{H}\tilde{\mathbf{B}}.\mathbf{K}_p.\mathbf{x}(k)$$

$$s(k+1) - s(k) = \mathbf{H}(\mathbf{A} - \mathbf{I}).\mathbf{x}(k) - \mathbf{HB}.\mathbf{K}_{er}.\mathbf{x}(k) + \mathbf{H}.\Delta\tilde{\mathbf{A}}.\mathbf{x}(k) - \mathbf{H}.\Delta\tilde{\mathbf{B}}.\mathbf{K}_{er}.\mathbf{x}(k) - \mathbf{H}\tilde{\mathbf{B}}.\mathbf{K}_p.\mathbf{x}(k)$$

$$s(k+1)-s(k)=\mathbf{HBK}_e.\mathbf{x}(k)-\mathbf{HB}.(\mathbf{K}_e+\mathbf{K}_r).\mathbf{x}(k)-\left(\mathbf{H\tilde{B}}.\mathbf{K}_p+\mathbf{H}.\Delta\mathbf{\tilde{B}}.\mathbf{K}_{er}-\mathbf{H}.\Delta\mathbf{\tilde{A}}\right).\mathbf{x}(k)$$

$$s(k+1)-s(k)=-\mathbf{HB}.\mathbf{K}_r.\mathbf{x}(k)-\left(\mathbf{H\tilde{B}}.\mathbf{K}_p+\mathbf{H}.\Delta\mathbf{\tilde{B}}.\mathbf{K}_{er}-\mathbf{H}.\Delta\mathbf{\tilde{A}}\right).\mathbf{x}(k)$$

or

$$s(k+1)-s(k)=-\rho.s(k)-\left(\mathbf{HB}.\mathbf{K}_p+\mathbf{H}.\Delta\mathbf{B}.\mathbf{K}_{er}-\mathbf{H}.\Delta\mathbf{A}\right).\mathbf{x}(k)$$

so

$$s(k+1)-s(k)=-\rho\, s(k)-\mathbf{R}\,\mathbf{x}(k), \qquad s(k+1)+s(k)=(2-\rho)\, s(k)-\mathbf{R}\,\mathbf{x}(k)$$

where

$$\mathbf{R}=\left(\mathbf{H\tilde{B}}\right)\mathbf{K}_p+\left(\mathbf{H}.\Delta\mathbf{\tilde{B}}\right)\mathbf{K}_{er}-\mathbf{H}.\Delta\mathbf{\tilde{A}} \tag{5.58}$$

thus

$$s^2(k+1)-s^2(k)=\{-\rho\, s(k)-\mathbf{R}\,\mathbf{x}(k)\}\,\{(2-\rho)\, s(k)-\mathbf{R}\,\mathbf{x}(k)\}$$

$$s^2(k+1)-s^2(k)=-\rho\,(2-\rho)\, s^2(k)+\left[-(2-\rho)+\rho\right] s(k)\mathbf{R}\,\mathbf{x}(k)+\left[\mathbf{R}\,\mathbf{x}(k)\right].\left[\mathbf{R}\,\mathbf{x}(k)\right]$$

$$s^2(k+1)-s^2(k)=-\rho\,(2-\rho)\, s^2(k)-2(1-\rho)\, s(k)\mathbf{R}\,\mathbf{x}(k)+\left[\mathbf{R}\,\mathbf{x}(k)\right].\left[\mathbf{R}\,\mathbf{x}(k)\right]$$

$$s^2(k+1)-s^2(k)=-\rho\,(2-\rho)\, s^2(k)-\left[2(1-\rho)\, s(k)-\mathbf{R}\,\mathbf{x}(k)\right]\mathbf{R}\,\mathbf{x}(k)$$

$$s^2(k+1)-s^2(k)=-\rho\,(2-\rho)\, s^2(k)-\left[2(1-\rho)\,\mathbf{H}\,\mathbf{x}(k)-\mathbf{R}\,\mathbf{x}(k)\right]\mathbf{R}\,\mathbf{x}(k)$$

or

$$s^2(k+1)-s^2(k)=-\rho\,(2-\rho)\, s^2(k)-\mathbf{x}^T(k)\left[2(1-\rho)\,\mathbf{H}^T-\mathbf{R}^T\right]\mathbf{R}\,\mathbf{x}(k)$$

To satisfy the sliding condition

$$|s(k+1)|<|s(k)| \;\Rightarrow\; s^2(k+1)<s^2(k) \;\Rightarrow\; s^2(k+1)-s^2(k)<0$$

we choose

$$\mathbf{x}^T(k)\left[2(1-\rho)\mathbf{H}^T\mathbf{R}-\mathbf{R}^T\mathbf{R}\right]\mathbf{x}(k)\geq 0$$

For a symmetric matrix, it can be shown that

$$\mathbf{Eig}\left\{\tfrac{1}{2}\left[2(1-\rho)\mathbf{H}^T\mathbf{R}-\mathbf{R}^T\mathbf{R}\right]+\tfrac{1}{2}\left[2(1-\rho)\mathbf{H}^T\mathbf{R}-\mathbf{R}^T\mathbf{R}\right]^T\right\}=\left[\alpha\pm\beta, \quad \mathbf{0}\right] \tag{5.59}$$

where

$$\alpha=\tfrac{1}{2}\left[\mathbf{R}\left(\gamma\mathbf{H}^T-\mathbf{R}^T\right)\right] \;\Rightarrow\; \sum_{i=1}^{n}\lambda_i=\mathbf{R}\left(\gamma\mathbf{H}^T-\mathbf{R}^T\right) \tag{5.60}$$

and

$$\gamma=2(1-\rho) \tag{5.61}$$

so a necessary condition for a positive-definite matrix is

$$\sum_{i=1}^{n}\left(\gamma\, h_i-r_i\right) r_i\geq 0 \tag{5.62}$$

we may choose

$$\left(\gamma\, h_i-r_i\right) r_i\geq 0, \quad i=1,\cdots,n \tag{5.63}$$

We will prove that

$$\gamma=2(1-\rho)>0 \tag{5.64}$$

(a) If $h_i>0$, then from Eq.(5.63) we have

$$0<\left(\mathbf{H\tilde{B}}\right)K_{pi}+\left(\mathbf{H}\;\Delta\mathbf{\tilde{B}}\right)K_{eri}-\mathbf{H}\;\Delta\mathbf{\tilde{A}}_{i(col)}<\gamma\;|h_i| \tag{5.65}$$

$$\mathbf{H}\left(\Delta\tilde{\mathbf{A}}_{i(col)}\right)-\left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri} < \left(\mathbf{H}\tilde{\mathbf{B}}\right)K_{pi} < \gamma\ |h_i| + \mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri}$$

- If $\mathbf{H}\tilde{\mathbf{B}} > 0$, then

$$\mathbf{H}\left(\Delta\tilde{\mathbf{A}}_{i(col)}\right)-\left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri} < \left|\mathbf{H}\tilde{\mathbf{B}}\right|K_{pi} < \gamma\ |h_i| + \mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri}$$

by Eq.(5.57.a), we have

$$\frac{\Delta_i}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \le K_{pi} \le \frac{\gamma\ |h_i| - \Delta_i}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \tag{5.66}$$

Assume that

$$\frac{\gamma\ |h_i| - \Delta_i}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \ge \frac{\Delta_i}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \tag{5.67}$$

$$\gamma\ |h_i| - \Delta_i \ge \Delta_i\ \frac{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \ \Rightarrow\ \gamma\ |h_i| \ge \Delta_i\left(1+\frac{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\right) \ \Rightarrow\ \gamma \ge \frac{\Delta_i}{|h_i|}\left(1+\frac{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\right)$$

$$\gamma \ge \max_i \frac{\Delta_i}{|h_i|}\left(1+\frac{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\right)$$

thus we obtain the condition in Eq.(5.57). By Eq.(5.57) and hence Eq.(5.67), we can choose

$$K_{pi} = \frac{\Delta_i}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} = \frac{\Delta_i}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\ \mathrm{sgn}(\mathbf{HB})\ \mathrm{sgn}(h_i) \tag{5.68.a}$$

under Assumption 5.1.

- If $\mathbf{H}\tilde{\mathbf{B}} < 0$, then

$$\mathbf{H}\left(\Delta\tilde{\mathbf{A}}_{i(col)}\right)-\left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri} < -\left|\mathbf{H}\tilde{\mathbf{B}}\right|K_{pi} < \gamma\ |h_i| + \mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri}$$

then under Assumption 5.1, we have

$$\frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} < -K_{pi} < \frac{\gamma\ |h_i| - \Delta}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \ \Rightarrow\ K_{pi} = -\frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} = \frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\ \mathrm{sgn}(\mathbf{HB})\ \mathrm{sgn}(h_i) \tag{5.68.b}$$

since Eq.(5.57) can be read as

$$\frac{\gamma\ |h_i| - \Delta_i}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \ge \frac{\Delta_i}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}$$

(b) If $h_i < 0$, then $-\gamma\ |h_i| < r_i < 0$

From Eq.(5.58), we have

$$-\gamma\ |h_i| + \mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri} < \left(\mathbf{H}\tilde{\mathbf{B}}\right)K_{pi} < +\mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri}$$

- If $\mathbf{H}\tilde{\mathbf{B}} > 0$, then we have

$$-\gamma\ |h_i| + \mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri} < \left|\mathbf{H}\tilde{\mathbf{B}}\right|K_{pi} < +\mathbf{H}\ \Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\ \Delta\tilde{\mathbf{B}}\right)K_{eri}$$

by the condition Eq.(5.57), under Assumption 5.1, we have

$$\frac{-\left(\gamma\,|h_i|-\Delta\right)}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \le K_{pi} \le \frac{-\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \;\Rightarrow\; K_{pi} = \frac{-\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} = \frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\operatorname{sgn}(\mathbf{HB})\operatorname{sgn}(h_i) \tag{5.68.c}$$

- If $\mathbf{H}\tilde{\mathbf{B}} < 0$, then

$$-\gamma\,|h_i| + \mathbf{H}\,\Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\,\Delta\tilde{\mathbf{B}}\right)K_{eri} < -\left|\mathbf{H}\tilde{\mathbf{B}}\right|K_{pi} < +\mathbf{H}\,\Delta\tilde{\mathbf{A}}_{i(col)} - \left(\mathbf{H}\,\Delta\tilde{\mathbf{B}}\right)K_{eri}$$

by the condition Eq.(5.57), under Assumption 5.1, we have

$$\frac{-\left(\gamma\,|h_i|-\Delta\right)}{\sup\left|\mathbf{H}\tilde{\mathbf{B}}\right|} < -K_{pi} < \frac{-\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} \;\Rightarrow\; K_{pi} = \frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|} = \frac{\Delta}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\operatorname{sgn}(\mathbf{HB})\operatorname{sgn}(h_i) \tag{5.68.d}$$

Q.E.D.

Remark 5.2: Sampling Condition in Discrete-Time SMC

- The conditions in Eqs.(5.50.d) and (5.57) are sampling conditions because they depend on **H**, **W** and **B** which in turn depend on the sampling rate.
- The sampling conditions may be too complicated to check, in particular Eq.(5.57). The implication is a faster sampling rate may be chosen for a lager uncertainty.

5.5.2.2. No Uncertainty

Under no uncertainties, we have the following corollary to design a linear discrete-time sliding mode controller

Corollary 5.1: Robust Discrete-Time Linear SMC Design under No Perturbation

A linear discrete-time sliding mode controller for a deterministic dynamical system can be determined by

$$u(k) = -\mathbf{K}.\mathbf{x}(k) = -\left(\mathbf{K}_e + \mathbf{K}_r\right).\mathbf{x}(k) \tag{5.69}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A}-\mathbf{I}) \tag{5.69.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{HB})^{-1}\rho.\mathbf{H} \tag{5.69.b}$$

and the sampling condition in Eq.(5.57) reduces to

$$(1-\rho) \ge 0 \tag{5.70}$$

with $\mathbf{x},\mathbf{B} \in \Re^{n\times 1}$, $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{H} \in \Re^{1\times n}$, $u,s \in \Re$

Proof

If there is no uncertainty, we have

$$\tilde{\mathbf{A}} = \mathbf{A},\quad \tilde{\mathbf{B}} = \mathbf{B},\quad \Delta\mathbf{A} = \mathbf{0},\quad \Delta\mathbf{B} = \mathbf{0}$$

then by Eq.(5.56.c), the perturbation control vanishes.

Q.E.D.

To design a linear discrete-time sliding mode controller, we must choose ρ. To do so, we propose the following theorem.

Theorem 5.7: Closed-Loop Eigenvalues composed of Hyperplane Eigenvalues and Sliding Margin

If a system has a discrete-time sliding margin ρ and a hyperplane-eigenvalue $\lambda_S \subset \operatorname{eig}\left[\mathbf{A}-\mathbf{B}(\mathbf{HB})^{-1}\mathbf{HA}\right]$, then the eigenvalues of the closed loop can be determined by

$$\lambda_C = \{1-\rho, \quad \lambda_H\} \in \Re^{1\times n} \tag{5.71}$$

Proof:

By definition of eigenvalues : $\operatorname{eig}(\mathbf{M}) = \left\{\lambda_i \,\middle|\, \det\left[\lambda_i \mathbf{I}-\mathbf{M}\right]=0\right\}$, from Eq.(5.69)

$$\mathbf{A}-\mathbf{BK} = \mathbf{A}-\mathbf{B}(\mathbf{HB})^{-1}\left[\mathbf{H}(\mathbf{A}-\mathbf{I})+\rho.\mathbf{H}\right] = \mathbf{A}-\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}.\left[\mathbf{A}-(1-\rho)\mathbf{I}\right]$$

To prove $(1-\rho)$ is one of the eigenvalues, we have to prove

$$\det\left[(1-\rho).\mathbf{I}-(\mathbf{A}-\mathbf{BK})\right]=0 \Rightarrow \det\left\{\left[(\mathbf{HB})^{-1}\mathbf{BH}-\mathbf{I}\right].\left(\mathbf{A}-(1-\rho).\mathbf{I}\right)\right\}=0 \tag{5.72}$$

By the properties of a determinant

$$|\mathbf{P}.\mathbf{Q}| = |\mathbf{P}|.|\mathbf{Q}| = |\mathbf{Q}|.|\mathbf{P}|, \qquad \mathbf{P},\mathbf{Q} \in \Re^{n\times n}$$

and

$$|\mathbf{I}_n+\mathbf{P}.\mathbf{Q}| = |\mathbf{I}_m+\mathbf{Q}.\mathbf{P}|, \qquad \mathbf{P} \in \Re^{n\times m}, \mathbf{Q} \in \Re^{m\times n}$$

then Eq.(5.72) is satisfied since

$$\left|(\mathbf{HB})^{-1}\mathbf{BH}-\mathbf{I}\right| = \left|(\mathbf{HB})^{-1}\mathbf{HB}-\mathbf{I}\right| = 0$$

Thus we can write

$$\left|\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A}-\mathbf{I})+\rho(\mathbf{HB})^{-1}\mathbf{H}\right| = (\lambda-1+\rho)\times\mathscr{P}^*\left(\lambda^{n-1}\right) \tag{5.73}$$

Eq.(5.73) holds for all ρ, so it must hold for $\rho=0$, then Eq.(5.73) becomes

$$\left|\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A}-\mathbf{I})\right| = (\lambda-1)\times\mathscr{P}^*\left(\lambda^{n-1}\right) \tag{5.74}$$

compared to Eq.(5.42), we have

$$\mathscr{P}^*\left(\lambda^{n-1}\right) \equiv \mathscr{P}\left(\lambda^{n-1}\right) \tag{5.75}$$

this proves that λ_H are the remaining eigenvalues. **Q.E.D**

Remark 5.3: Sampling Condition equivalent to Stability Criterion with Unit Circle

Since $(1-\rho)$ is a closed loop eigenvalue, to stabilize the system we must choose within the unit circle

$$|1-\rho| < 1 \tag{5.76}$$

hence the sampling condition in Eq.(5.70) is consistent with this condition. So the condition Eq.(5.70) is also the condition of stability. In terms of a continuous-time sliding margin δ, we have the following relationship

$$\rho = 1-e^{-T\delta} \tag{5.77}$$

The sampling condition Eq.(5.57) and hence Eq.(5.70) can be relaxed with a fast sampling rate, i.e. the faster sampling rate is the easier this sampling condition satisfies.

5.5.2.3. Matched Uncertainty

To determine the performance of a robust control design, we propose the following theorem

Theorem 5.8: Robust Discrete-Time Linear SMC Design under Matched Uncertainty

Consider an uncertain dynamical linear discrete-time system

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x}(k) + \mathbf{B}.u(k), \qquad \left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}$$

with a hyperplane

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B}$$

If $\Delta\mathbf{A}$ satisfies the matching uncertainties condition then the system-eigenvalues at the upper boundary

$$\lambda_{C,B} = \{1-\rho, \quad \lambda_H\} \tag{5.78}$$

where

$$\mathbf{x}.\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad \lambda_H \in \Re_{(1)}^{1\times(n-1)}, \quad u, s \in \Re, \quad \delta \in \Re_{(1)}$$

with

$$\Re_{(1)}^{1\times(n-1)}, \quad \Re_{(1)}: \text{unit circle}$$

Proof

From Eq.(5.56.c) with $\Delta\mathbf{B} = \mathbf{0}$

$$\mathbf{K}_p = \frac{\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}.\left|\mathbf{K}_{er}\right|\right)}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\operatorname{sgn}(\mathbf{HB}) = \frac{\mathbf{H}.\Delta\mathbf{A}}{|\mathbf{HB}|}\operatorname{sgn}(\mathbf{HB}) = (\mathbf{HB})^{-1}\mathbf{H}.\Delta\mathbf{A} \tag{5.79}$$

Since all the elements of $\Delta\mathbf{A}$ are positive constant, from the matching uncertainties condition

$$\Delta\mathbf{A} = \mathbf{B}.\mathbf{v}, \qquad \mathbf{v} \in \Re^{1\times n}$$

so Eq.(5.79) becomes

$$\mathbf{K}_p = (\mathbf{HB})^{-1}\mathbf{H}.\mathbf{B}.\mathbf{v} = \mathbf{v}$$

Thus the system-eigenvalues at the upper boundary, $\tilde{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}$, are

$$\lambda_{C2} = \operatorname{eig}(\mathbf{A}_2 - \mathbf{B}.\mathbf{K}) = \operatorname{eig}\left[\mathbf{A} + \Delta\mathbf{A} - \mathbf{B}.\left(\mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p\right)\right] = \operatorname{eig}\left[\mathbf{A} + \mathbf{B}.\mathbf{v} - \mathbf{B}.\left(\mathbf{K}_e + \mathbf{K}_r + \mathbf{v}\right)\right] = \operatorname{eig}\left[\mathbf{A} - \mathbf{B}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right]$$

Recall that, under no perturbation we have

$$\operatorname{eig}\left(\mathbf{A} - \mathbf{B}.\mathbf{K}_{er}\right) = \{1-\rho, \quad \lambda_H\}$$

so

$$\lambda_{C2} = \operatorname{eig}\left[\mathbf{A} - \mathbf{B}.\left(\mathbf{K}_e + \mathbf{K}_r\right)\right] = \{1-\rho, \quad \lambda_H\}$$

Q.E.D.

Theorem 5.8 has shown that system eigenvalues of an uncertain dynamical system at upper boundary $(\mathbf{A} + \Delta\mathbf{A})$ are equal to those of a deterministic dynamical system. We propose the following design rule to move these eigenvalues inside the bounds by shifting down the nominal values of the system matrix.

Proposition 5.3: Robust Discrete-Time Linear SMC Design Rule

Theorem 5.8 is valid for matched uncertain systems and **H** has all element the same polarity. In general, choosing the nominal system matrices as

$$\mathbf{A} \rightarrow \mathbf{A} + \tfrac{1}{2}\Delta\mathbf{A}, \qquad \mathbf{B} \rightarrow \mathbf{B} + \tfrac{1}{2}\Delta\mathbf{B}$$

in designing a hyperplane may produce a more robust controller.

5.5.3. Robust Discrete-Time Linear SMC under Uncertainties and Disturbances

From Theorem 5.4 and 5.5, we have the following corollary to design a robust SMC for systems under uncertainty and disturbance.

Corollary 5.2: Robust Discrete-Time Linear SMC Design under Uncertainty and Disturbance

Consider a discrete-time system under uncertainty and disturbances

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) + \mathbf{W}.v(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k) + \mathbf{W}.v(k), \qquad |v(k) \leq \bar{v}| \tag{5.80}$$

with a hyperplane

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \leq \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \leq \Delta\mathbf{B}$$

with

$$\mathbf{x} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B}, \Delta\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re$$

then, under Assumption 5.1, a robust discrete-time linear sliding mode control function can be determined by

$$\underline{\underline{u(k) = -\mathbf{K}.\mathbf{x}(k), \qquad \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r + \mathbf{K}_p}} \tag{5.81}$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{H}\mathbf{B})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}) \tag{5.81.a}$$

- reaching control

$$\mathbf{K}_r = (\mathbf{H}\mathbf{B})^{-1}\rho.\mathbf{H}, \quad \rho = 1 - e^{-T\delta} \tag{5.81.b}$$

- perturbation control

$$\mathbf{K}_p = \mathbf{K}_u + \mathbf{K}_d \tag{5.81.c}$$

* uncertainty control

$$\mathbf{K}_u = \left[\frac{\mathbf{H}\left(\Delta\mathbf{A} + \Delta\mathbf{B}\left|\mathbf{K}_e + \mathbf{K}_r\right|\right)}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|}\right] \otimes \operatorname{sgn}(\mathbf{H}\mathbf{B}) \tag{5.81.d}$$

* disturbance control

$$\mathbf{K}_d = \left(\mathbf{H}\bar{\mathbf{B}}\right)^{-1}\sup|\mathbf{H}\mathbf{W}|.\bar{v}.\mathbf{H} \tag{5.81.e}$$

if the following sampling conditions is satisfied

$$(1-\delta) \geq \left(1 + \frac{\sup|\mathbf{H}\mathbf{B}|}{\inf|\mathbf{H}\mathbf{B}|}\right)\max_i \frac{\bar{\Delta}_i}{2.|h_i|}, \qquad \bar{\Delta}_i = |\mathbf{H}|.\left(\Delta\mathbf{A}_{i(col)} + \Delta\mathbf{B}\,|K_{eri}|\right) \tag{5.81.f}$$

and

$$0 < \rho + \rho_p \sup|\mathbf{H}\mathbf{W}|.\bar{v} < 1 \tag{5.81.g}$$

and if the following disturbance condition is satisfied

$$|s(k)| > \sup\left(\frac{\rho_p \sup|\mathbf{HW}|.\bar{v}}{\rho + \rho_p \sup|\mathbf{HW}|.\bar{v}}, \frac{\rho_p \sup|\mathbf{HW}|.\bar{v}}{1 - \left(\rho + \rho_p \sup|\mathbf{HW}|.\bar{v}\right)}\right) \quad (5.81.h)$$

Proof

Eqs(5.81.a), (5.81.b) and (5.81.d) can be determined using Theorem 5.5. Since the uncertainty control can tackle uncertainty, Eq.(5.80) can be read as

$$\mathbf{x}(k+1) = \mathbf{A}.\mathbf{x}(k) + \mathbf{B}.u(k) + \mathbf{W}.v(k) \quad (5.82)$$

the Eq.(5.81.e) can be determined by Eq.(5.82) using Theorem 5.4.

Q.E.D.

Based on the results on the robust VSS controller design Eq.(4.7) in Chapter and on the results on the robust discrete-time linear SMC Eq.(5.80) above, we have the following proposition for the robust discrete-time VSS controller design

Proposition 5.4: Robust Discrete-Time VSS Controller Design under Uncertainty and Disturbance

For a linear system under perturbations (uncertainties and disturbances)

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) + \mathbf{W}.v(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k) + \mathbf{W}.v(k), \qquad |v(k) \le \bar{v}|$$

and a hyperplane

$$s(k) = \mathbf{H}.\mathbf{x}(k)$$

with

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| = \Delta\mathbf{B}, \qquad |v| = \bar{v}$$

then, under Assumption 4.1, there exists a constant $\delta > 0$ for a *VSS control function* to be determined as

$$\underline{\underline{u = u_e + u_r + u_p}} \quad (5.83)$$

where

- equivalent control

$$\mathbf{K}_e = (\mathbf{HB})^{-1}\mathbf{H}(\mathbf{A} - \mathbf{I}) \quad (5.83.a)$$

- reaching control

$$u_r = -(\mathbf{HB})^{-1}\mathbf{K}_r.|\mathbf{x}|.\operatorname{sgn}(s), \qquad \mathbf{K}_r = \rho.|\mathbf{H}|, \qquad \rho = 1 - e^{-T\delta} \quad (5.83.b)$$

- perturbations control

$$u_p = -\left(K_{p0} + \mathbf{K}_p.|\mathbf{x}|\right).\operatorname{sgn}(s), \qquad K_{p0} = \frac{|\mathbf{H}.\mathbf{W}|\bar{v}}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\operatorname{sgn}(\mathbf{HB})}, \qquad \mathbf{K}_p = \frac{|\mathbf{H}|.\Delta\mathbf{A}}{\inf\left|\mathbf{H}\tilde{\mathbf{B}}\right|.\operatorname{sgn}(\mathbf{HB})}, \qquad |\mathbf{x}| = \left[|x_i|\right]$$

(5.83.c)

with

$$\mathbf{x}, \mathbf{B}, \Delta\mathbf{B}, \mathbf{W} \in \Re^{n\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s, v \in \Re, \quad \delta \in \Re_{(+)}$$

5.6. A ROBUST INTEGRAL DISCRETE-TIME LINEAR SMC DESIGN

A SMC is based on a state-space model, an I-action may be required to eliminate a steady-state error. We have the following corollary to design a robust integral discrete-time SMC using Theorem 5.5.

Theorem 5.9: Robust Discrete-Time Linear SMC Design under Uncertainty

Consider an uncertain dynamical linear system system

$$\begin{cases} \mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k), & \left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \quad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B} \\ y(k) = \mathbf{C}.\mathbf{x}(k) \end{cases}$$

and its augmented-order system with a reference input of r

$$\mathbf{x}_i(k+1) = \tilde{\mathbf{A}}_i.\mathbf{x}_i(k) + \tilde{\mathbf{B}}_i.u(k) \tag{5.84}$$

where

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} 1 & \mathbf{C}\tilde{\mathbf{A}} \\ \mathbf{0}_{n\times1} & \tilde{\mathbf{A}} \end{bmatrix}, \quad \tilde{\mathbf{B}}_i = \begin{bmatrix} \mathbf{C}\tilde{\mathbf{B}} \\ \tilde{\mathbf{B}} \end{bmatrix} \tag{5.84.a}$$

then a robust integral discrete-time SMC with a hyperplane $\tilde{\mathbf{H}}$ can be determined by

$$\underline{\underline{u(k) = u_*(k) + \tilde{u}_i(k)}} \tag{5.85}$$

where

$$u_*(k) = \left(\mathbf{H}_i\mathbf{B}_i\right)^{-1} h_{i,1}\ r, \qquad \mathbf{H}_i = \begin{bmatrix} h_{i,1} & \cdots & h_{i,n+1} \end{bmatrix} \tag{5.85.a}$$

and

$u_i(k)$ is a robust SMC for Eq.(5.84) and can be determined by Theorem 5.5.

with

$\mathbf{x}, \mathbf{B} \in \Re^{n\times1}, \quad \mathbf{A} \in \Re^{n\times n}, \quad \mathbf{C} \in \Re^{1\times n}, \quad u, \tilde{u}, \bar{u}_*, y, r \in \Re$: sliding margin.

Proof

With an input reference of r, define an integral as

$$x_0(t) = \int_0^t (y - r)\, dt$$

then its discretization is

$$x_0(z) = \frac{y(z) - r(z)}{1 - z^{-1}} \quad \Rightarrow \quad \left(1 - z^{-1}\right).x_0(z) = y(z) - r(z)$$

or

$$x_0(k+1) - x_0(k) = y(k+1) - r(k+1)$$

thus

$$x_0(k+1) - x_0(k) = \mathbf{C}.\mathbf{x}(k+1) - r(k+1) = \mathbf{C}.\left[\tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k)\right] - r(k+1) = \mathbf{C}\tilde{\mathbf{A}}.\mathbf{x}(k) + \mathbf{C}\tilde{\mathbf{B}}.u(k) - r(k+1)$$

or

$$x_0(k+1) = x_0(k) + \mathbf{C}\tilde{\mathbf{A}}.\mathbf{x}(k) + \mathbf{C}\tilde{\mathbf{B}}.u(k) - r(k+1)$$

so we obtain the following augmented system

$$\mathbf{x}_i(k+1) = \begin{bmatrix} x_0(k+1) \\ \mathbf{x}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{C}\tilde{\mathbf{A}} \\ \mathbf{0}_{n\times1} & \tilde{\mathbf{A}} \end{bmatrix}.\begin{bmatrix} x_0(k) \\ \mathbf{x}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{C}\tilde{\mathbf{B}} \\ \tilde{\mathbf{B}} \end{bmatrix}.u(k) + \begin{bmatrix} -1 \\ \mathbf{0} \end{bmatrix}.r(k+1)$$

from Eq.(5.84.a), we have

$$\dot{\mathbf{x}}_i = \tilde{\mathbf{A}}_i.\mathbf{x}_i + \tilde{\mathbf{B}}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \left(\mathbf{A}_i + \Delta\tilde{\mathbf{A}}_i\right).\tilde{\mathbf{x}} + \left(\mathbf{B}_i + \Delta\tilde{\mathbf{B}}_i\right).u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r \tag{5.86}$$

where

$$r = r(k+1)$$

since the uncertainty component of u can tackle any uncertainty $\Delta\tilde{\mathbf{B}}_i$, Eq.(5.85) is still hold for $\Delta\tilde{\mathbf{B}}_i = \mathbf{0}$ so it can be read as

$$\dot{\mathbf{x}}_i = \left(\mathbf{A}_i + \Delta\tilde{\mathbf{A}}_i\right).\tilde{\mathbf{x}} + \mathbf{B}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \tilde{\mathbf{A}}_i.\tilde{\mathbf{x}} + \mathbf{B}_i.u + \begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r \tag{5.86.a}$$

thus

$$s_i(k+1) = \mathbf{H}_i.\mathbf{x}_i(k+1) = \mathbf{H}_i\tilde{\mathbf{A}}_i.\tilde{\mathbf{x}}(k) + \mathbf{H}_i.\mathbf{B}_i.u(k) + \mathbf{H}_i.\begin{bmatrix} -1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}.r = \mathbf{H}_i\tilde{\mathbf{A}}_i\tilde{\mathbf{x}} + \mathbf{H}_i\mathbf{B}_i u - h_{i,1}\, r$$

by Eq.(5.85.a), we have

$$s_i(k+1) = \mathbf{H}_i\tilde{\mathbf{A}}_i.\mathbf{x}_i(k) + \mathbf{H}_i\mathbf{B}_i.u(k) - \mathbf{H}_i\mathbf{B}_i.\bar{u}_*(k) = \mathbf{H}_i\tilde{\mathbf{A}}_i.\mathbf{x}_i(k) + \mathbf{H}_i\mathbf{B}_i.\tilde{u}_i(k)$$

or

$$\tilde{s}_i(k+1) = \mathbf{H}_i\left[\tilde{\mathbf{A}}_i.\mathbf{x}_i(k) + \mathbf{B}_i.\tilde{u}_i(k)\right] \tag{5.87}$$

since $\tilde{u}_i$ is a robust SMC for Eq.(5.84), the sliding condition $\tilde{s}_i^2(k+1) - \tilde{s}_i^2(k) \le 0$ is satisfied using Eq.(5.87)

Q.E.D.

5.7. A NEW ROBUST DISCRETE-TIME LINEAR SLIDING-MODE OBSERVER DESIGN

As in the state-space observer design where the estimator response is 3 to 10 faster than that of the controller response, to design a robust observer, we propose the following theorem

Theorem 5.10: Robust Discrete-Time Linear Sliding-Mode Observer Design under Uncertainty

Consider a system state equation

$$\begin{cases} \mathbf{x}(k+1) = \tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k) \\ \mathbf{y}(k) = \tilde{\mathbf{C}}\,\mathbf{x}(k) = \left(\mathbf{C} + \Delta\tilde{\mathbf{C}}\right)\mathbf{x}(k): \text{ available output} \end{cases} \tag{5.88}$$

where

$$\left|\Delta\tilde{\mathbf{A}}\right| \le \Delta\mathbf{A}, \qquad \left|\Delta\tilde{\mathbf{B}}\right| \le \Delta\mathbf{B}$$

with

$$\mathbf{x} \in \Re^{n\times 1}, \quad \mathbf{y} \in \Re^{p\times 1}, \quad \mathbf{A}, \Delta\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B}, \Delta\mathbf{B} \in \Re^{n\times 1}, \quad \mathbf{C}, \Delta\mathbf{C} \in \Re^{p\times n}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re$$

if the system is observable, then a robust SMO can be found from

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}u(k) + \mathbf{L}.\left[\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right], \qquad \hat{\mathbf{y}}(k) = \tilde{\mathbf{C}}.\hat{\mathbf{x}}(k) \tag{5.89}$$

where

$$\underline{\underline{\mathbf{L} = \mathbf{L}_e + \mathbf{L}_r + \mathbf{L}_p}} \tag{5.90}$$

with

$$\mathbf{L}_e = (\mathbf{A} - \mathbf{I})\mathbf{H}_o'\left(\mathbf{C}\mathbf{H}_o'\right)^{-1} \tag{5.91.a}$$

$$\mathbf{L}_r = \rho_o \mathbf{H}_o'\left(\bar{\mathbf{C}}\mathbf{H}_o'\right)^{-1}, \qquad \rho_o = 1 - e^{-T\delta_o}, \qquad \delta_o = (3..10) \times \delta \tag{5.91.b}$$

$$\mathbf{L}_p = \operatorname{sgn}(\mathbf{H}_o) \otimes \frac{\left(\Delta\mathbf{A}' + |\mathbf{L}_e + \mathbf{L}_r|.\Delta\mathbf{C}\right).|\mathbf{H}_o'|}{\inf|\mathbf{H}_o\tilde{\mathbf{C}}'|.\operatorname{sgn}(\mathbf{H}_o\mathbf{C}')} \tag{5.91.c}$$

$\mathbf{H}_o$ is determined by the eigenvalue allocation with the observer hyperplane-eigenvalues

$$\lambda_{Ho} = (3..10) \times \lambda_{Hc} \quad \Rightarrow \quad \mathbf{H}_o = \operatorname{hyper}\left(\mathbf{A}', \mathbf{C}', \lambda_{H_o}\right)$$

Proof

Since **L** will be determined under uncertainty, Eq.(5.89) can be perturbed to read as

$$\hat{\mathbf{x}}(k+1) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\hat{\mathbf{x}}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u(k) + \mathbf{L}.\left[\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right]$$

then an error equation $\mathbf{e}(k+1) = \hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1)$ can be determined by

$$\mathbf{e}(k+1) = \hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1) = \tilde{\mathbf{A}}.\hat{\mathbf{x}}(k) + \tilde{\mathbf{B}}.u(k) + \mathbf{L}\tilde{\mathbf{C}}\left[\mathbf{x}(k) - \hat{\mathbf{x}}(k)\right] - \left[\tilde{\mathbf{A}}.\mathbf{x}(k) + \tilde{\mathbf{B}}.u(k)\right] = \left(\tilde{\mathbf{A}} - \tilde{\mathbf{L}}\tilde{\mathbf{C}}\right).\mathbf{e}(k) \tag{5.92}$$

To determine **L**, we will find a function *v* to nullify the error by the following equation

$$\mathbf{e}(k+1) = \tilde{\mathbf{A}}'\mathbf{e}(k) + \tilde{\mathbf{C}}'v(k) \tag{5.93}$$

By Theorem 5.5, we can find $v(k) = -\mathbf{L}'\ \mathbf{e}(k)$ (5.94)

where $\mathbf{L}'$ is determined by Eqs.(5.93), so Eq.(5.92) can be read

$$\mathbf{e}(k+1) = \tilde{\mathbf{A}}'\mathbf{e}(k) - \tilde{\mathbf{C}}'\tilde{\mathbf{L}}'\mathbf{e}(k) = \left(\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}}\right)'\ \mathbf{e}(k) \tag{5.95}$$

we have the following mapping to complete the proof

$$\tilde{\mathbf{A}} \to \tilde{\mathbf{A}}', \quad \tilde{\mathbf{B}} \to \tilde{\mathbf{C}}', \quad \mathbf{H} \to \mathbf{H}_o, \quad \mathbf{K} \to \mathbf{L}' \tag{5.96}$$

so Eqs.(5.90) have been proved.

Q.E.D.

5.8. NUMERICAL EXAMPLES

Some original designs from the literature will be used to compare with the proposed design. Sliding parameters are the same as in the previous chapter, if available; however, reaching parameters may be changed, if necessary, to conform with the sliding-mode design rule.

Remark 5.4: Summary of Robust Discrete SMC Designs

In the numerical examples below, and in this work generally, hyperplane eigenvalues will be chosen at the same unique value for simplicity. Different multiple values may be attempted to compromise between the response speed and overshoot.

- If continuous-time sliding-mode controllers (continuous and discontinuous SMC) are available, then Proposition 5.2 can be used to obtain discrete-time sliding-mode controllers. However, discrete-time sliding-mode observers must be designed in discrete-time domain;
- Nominal model can be determined using proposition 5.3 in designing a hyperplane (Chapter 2)
- A sampling rate can be chosen on the basis of Proposition 5.1 and Remark 5.2;
- Sliding margin δ is chosen on the basis of Proposition 3.1, and the mapping is $\rho = 1 - e^{-T\delta}$;
- New Robust Discrete-Time Discontinuous SMC functions are computed by Theorem 5.4 or Proposition 5.4 where the latter is a discretization of Theorem 4.3;

- New Robust Discrete-Time Linear (Continuous) SMC functions are computed by Corollary 5.2 deduced from Theorems 5.5 and 5.6 for a general case under uncertainty and disturbance where a special case without perturbation (uncertainty and/or disturbance) will make the corresponding control component(s) vanish;
- Robust Discrete-Time Integral SMC functions are computed by Theorem 5.9;
- New discrete-time robust sliding-mode observers are computed by Theorem 5.10

5.8.1 Example 5.1: No Uncertainty

Consider a SISO discrete-time state equation in Furuta 1990

$$\mathbf{A} = \begin{bmatrix} 0.9953, & 0.0905 \\ -0.0905, & 0.8144 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.0047 \\ 0.0905 \end{bmatrix}, \quad T_s = 0.1\,\text{sec}$$

5.8.1.1. Original Design

The original control function is proposed in Furuta1990 as

$$u(k) = -(\mathbf{K}_e + \mathbf{K}_r)\,\mathbf{x}(k), \qquad \mathbf{K}_e = [1, \quad 1]$$

where

$$\delta = 5 \quad \Rightarrow \quad k_{ri} = \begin{cases} \delta\,\text{sgn}\{\mathbf{HB}s(k)x_i\}, & \text{if } |s(k)| > \beta, \quad \beta = \tfrac{1}{2}|\mathbf{HB}|\left(\delta \sum_{i=1}^{n} |x_i|\right) \\ 0, & \text{else} \end{cases}$$

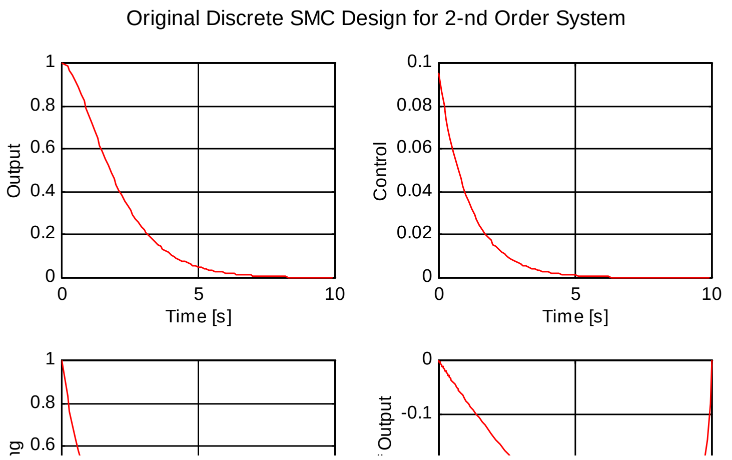


Fig. 5.8: Original Discrete SMC Design in Example 5.1

5.8.1.2. New Design

Choose hyperplane-eigenvalues as

$$p_1 = -0.8 \;\Rightarrow \mathbf{H} = [8.4899, \quad 10.6127]$$

- Linear SMC

Choose the reaching dynamics 1.5 times faster than the sliding dynamics, then Corollary 5.2 yields

$$u(k) = -\mathbf{K}.\mathbf{x}(k), \qquad \mathbf{K} = [0.0772, \quad 0.1161]$$

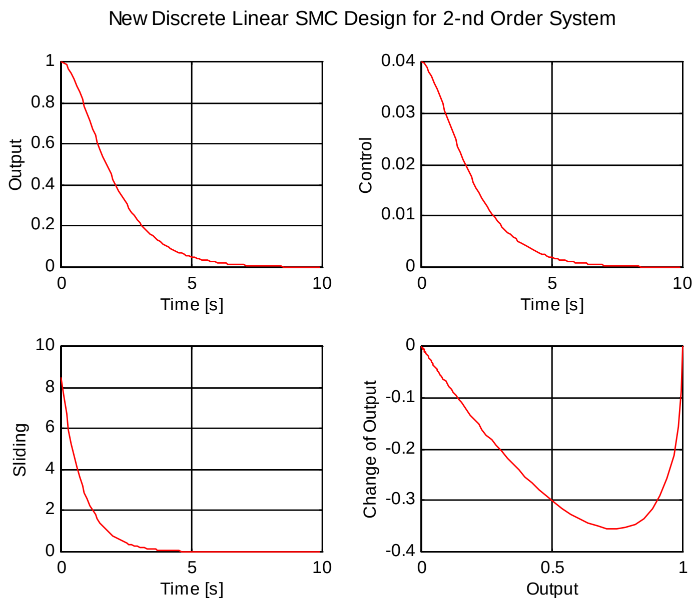


Fig. 5.9: New Discrete Linear SMC Design in Example 5.1

- Discontinuous SMC

Choose $\delta = 0.4$, then Theorem 5.4 yields

$$\mathbf{K}_r = [-0.4 \quad 0.4]$$

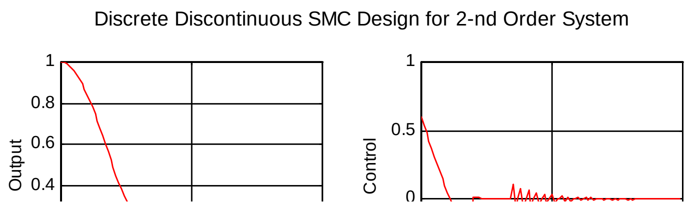


Fig. 5.10: Discrete Discontinuous SMC in Example 5.1

5.8.2. Example 5.2: Matched Uncertainty

Consider the following system from Sarpturk *et al.* 1987

$$\mathbf{x}(k+1) = \mathbf{A}.\mathbf{x}(k) + \mathbf{B}.[u(k) + p(k)]$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 \\ -0.055 & 1.2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.055 \end{bmatrix}, \quad p(k) = \sin(0.1k\pi)$$

Note that this form of perturbation has also appeared in Spurgeon 1992.

5.8.2.1. Original Design

The proposed control function from the original design is

$$u(k) = -\psi_1 x_1(k) - \psi_2 x_2(k)$$

with

$$\psi_1 = \begin{cases} +138, & if\ s(k).x_1(k) > 0 \\ -140, & if\ s(k).x_1(k) < 0 \end{cases}, \quad \psi_2 = \begin{cases} +36, & if\ s(k).x_2(k) > 0 \\ +14, & if\ s(k).x_2(k) < 0 \end{cases}$$

and the hyperplane is

$$s(k) = \mathbf{H}.\mathbf{x}(k), \quad \mathbf{H} = [20, \quad 1]$$

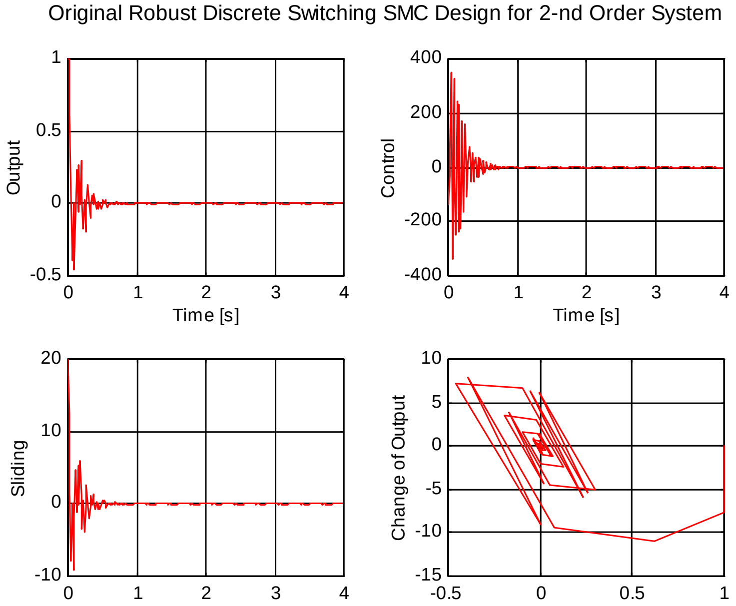


Fig. 5.11: Original Robust Discrete Switching SMC Design under Matched Uncertainty in Example 5.2.

5.8.2.2. New Design

Choose $\lambda_H = [-3]$, then

$$\mathbf{H} = [50.65 \quad 18.18]$$

thus

$$\mathbf{K}_e = [-1 \quad 4.4885]$$

- Robust Linear SMC

  Choose $\delta = 9$ and $\delta_p = 50$, then Corollary 5.2 yields

$$\mathbf{K} = [10.6703 \quad 16.9391]$$

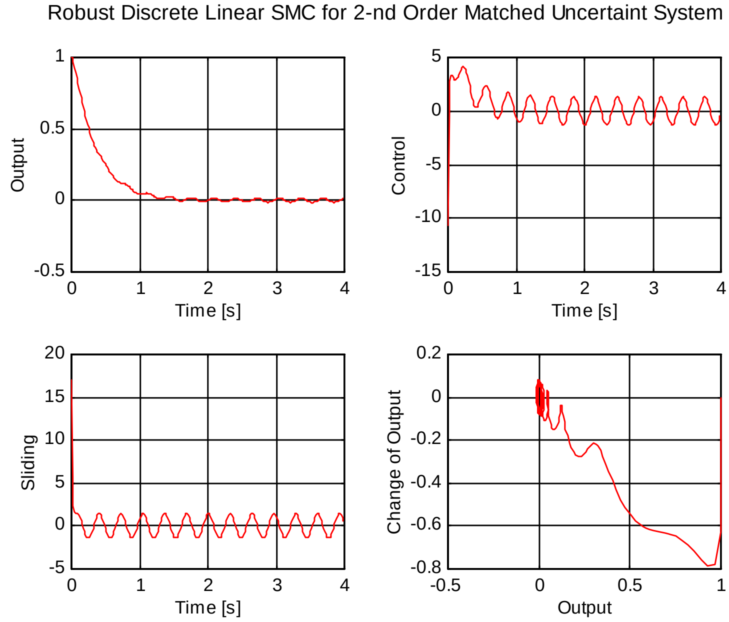


Fig. 5.12: Robust Linear SMC under Matched Uncertainty in Example 5.2.

- Robust Discontinuous SMC

  Choose $\delta = 3$, then Theorem 5.4 yields

$$\mathbf{K}_r = [3 \quad 3]$$

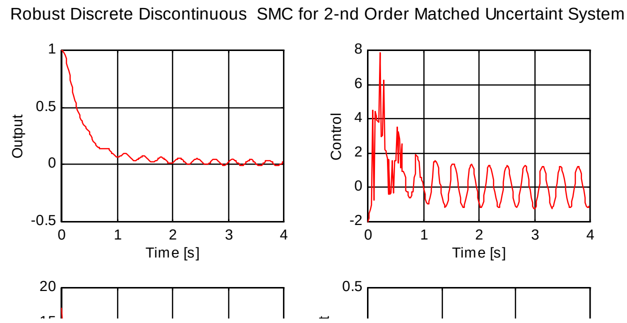


Fig. 5.13: Robust Discrete Discontinuous SMC under Matched Uncertainty in Example 5.2

5.8.3. Example 5.3: Integral Discrete-Time SMC

Consider a SISO discrete-time state equation in Chan 1991

$$\mathbf{A} = \begin{bmatrix} 0.99532116, & 0.0904837418 \\ -0.0904827418, & 0.8142533676 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.00467884016 \\ 0.090904837418 \end{bmatrix}, \quad T_s = 0.1 \text{ sec}$$

5.8.3.1. Original Design

The original control function is proposed in Chan 1991 as

$$u(k) = -\psi_0\, s(k) - \left(\mathbf{K}_e + \mathbf{K}_r\right)\mathbf{x}(k), \qquad \psi_0 = 8, \qquad \mathbf{K}_e = \left[-0.9958, \quad -0.0477\right]$$

where

$$\left.\begin{array}{l}\delta = 3\\ \rho = 0.4\end{array}\right\} \Rightarrow \quad k_{ri} = \begin{cases}\dfrac{\delta}{1-\rho}\,\mathrm{sgn}\left\{\mathbf{HB}s(k)x_i\right\}, & \text{if } \left|s(k)\right| > \beta, \quad \beta = \tfrac{1}{2}\left|\mathbf{HB}\right|\left(\delta\sum_{i=1}^{n}\left|x_i\right|\right)\\ 0, & \text{else}\end{cases}$$

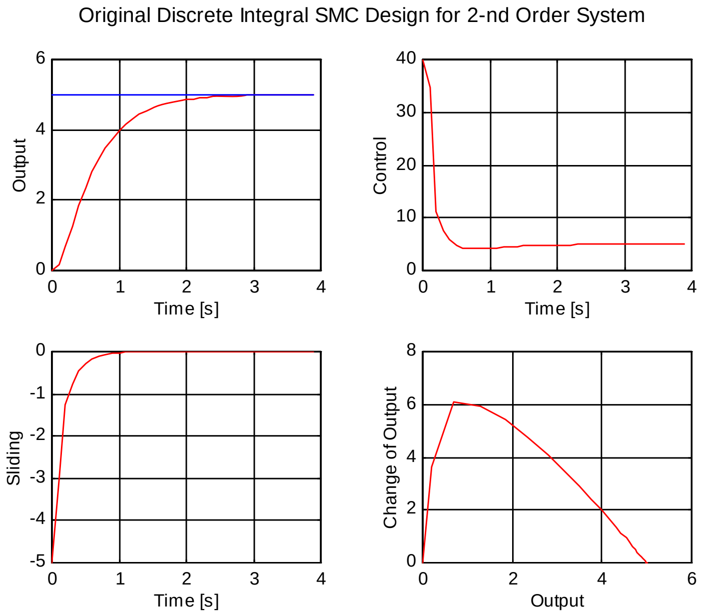


Fig. 5.14: Original Discrete Integral SMC Design for Servo-System in Example 5.3.

5.8.3.2. New Design

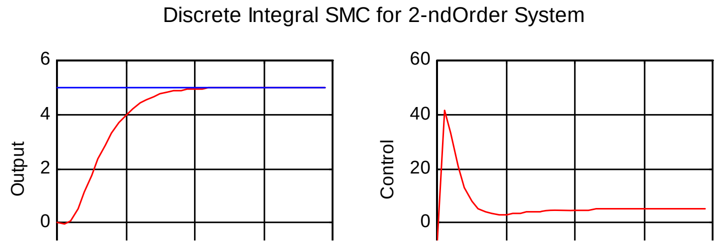


Fig. 5.15: Discrete Integral SMC for Servo-System in Example 5.2.

Choose $\lambda_H = [-3.5 \quad -3.5]$ for the augmented system defined by Eq.(5.84), then

$$\mathbf{H}_i = [9.8592 \quad 50.4187 \quad 7.9119]$$

and the reaching dynamics 5 times faster than the sliding dynamics, then Theorem 5.9 yields

$$u(k) = -h_{i,1}r - \mathbf{K}_i\mathbf{x}_i, \quad \mathbf{K}_i = [7.9229 \quad 50.2498 \quad 10.4972], \quad \mathbf{x}_i = \begin{bmatrix} \sum[x_1(k) - r] \\ \mathbf{x} \end{bmatrix}$$

5.8.4. Example 5.4: Deterministic System

Consider a linear 4-th order system (Sivaramakrishnan et al. 1984) as in Example 4.1

$$\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c u, \qquad \mathbf{A}_c = \begin{bmatrix} -0.0500, & 6.0000, & 0, & 0 \\ 0, & -3.3330, & 3.3330, & 0 \\ -5.2080, & 0, & -12.5000, & -12.5000 \\ 0.6000, & 0, & 0, & 0 \end{bmatrix}, \qquad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 12.5000 \\ 0 \end{bmatrix}$$

Choose $\lambda_H = [-6, \quad -6, \quad -6]$, then

$$\mathbf{H} = [0.4285 \quad 0.3508 \quad 0.0800 \quad 1.4401]$$

thus

$$\mathbf{K}_e = [0.4260 \quad 1.4014 \quad 0.1694 \quad -1.0000]$$

and the sliding-eigenvalues are unchanged

$$\lambda_S = [-6, \quad -6, \quad -6]$$

and choose the sampling time as

$$T_s = 0.01 \,\text{sec}$$

since it is about 17 times faster than the desired dynamics λ_H, so the corresponding discrete-time system is

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,u(k), \qquad \mathbf{A} = \begin{bmatrix} 0.9995 & 0.0590 & 0.0009 & 0 \\ -0.0008 & 0.9672 & 0.0308 & -0.0020 \\ -0.0493 & -0.0015 & 0.8825 & -0.1175 \\ 0.0060 & 0.0002 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0.0020 \\ 0.1175 \\ 0 \end{bmatrix}$$

Remark 5.5: Comparison of *Continuous* and *Discrete* Hyperplane

Using the above discrete model to design a hyperplane, we obtain

$$\tilde{\mathbf{H}} = [42.3857 \quad 34.7105 \quad 7.9120 \quad 142.4628]$$

this *discrete* hyperplane differs by the above *continuous* one only by a scale factor of 100, so they are essentially the same (Section 2.5).

- Discrete Linear SMC

Based on the robust practical design rule, choose $\delta = 18$, then Corollary 5.2 yields

$$\mathbf{K}_r = [7.7122, \quad 6.3152, \quad 1.4400, \quad 25.9226]$$

thus

$$\mathbf{K} = [8.1382, \quad 7.7165, \quad 1.6094, \quad 24.9226]$$

and closed-loop system-eigenvalues are

$$\lambda_C = [-6, \ -6, \ -6, \ -18]$$

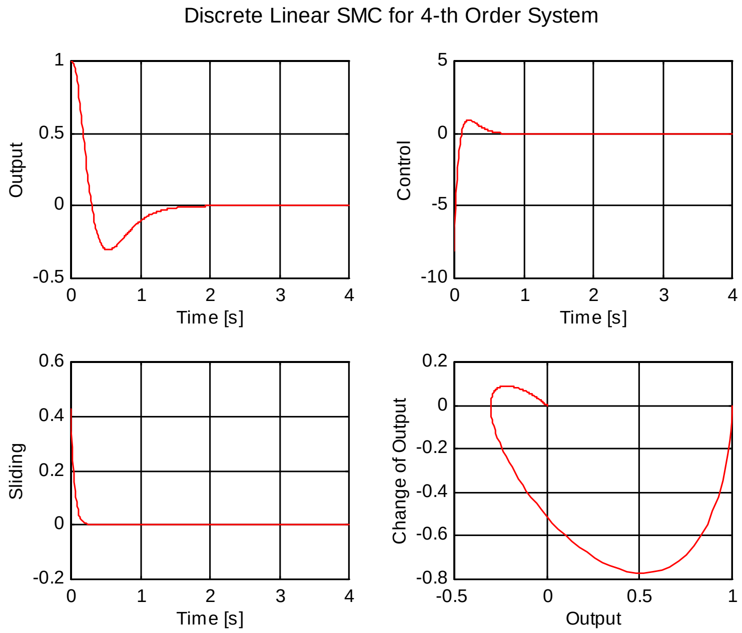


Fig. 5.16: Discrete Linear SMC for 4-th Order System in Example 5.4.

- Discrete Switching SMC

Choose $\delta = 10$, then Proposition 5.4 yields

$$\mathbf{K}_r = \begin{bmatrix} 4.2845 & 3.5084 & 0.8 & 14.4014 \end{bmatrix}$$

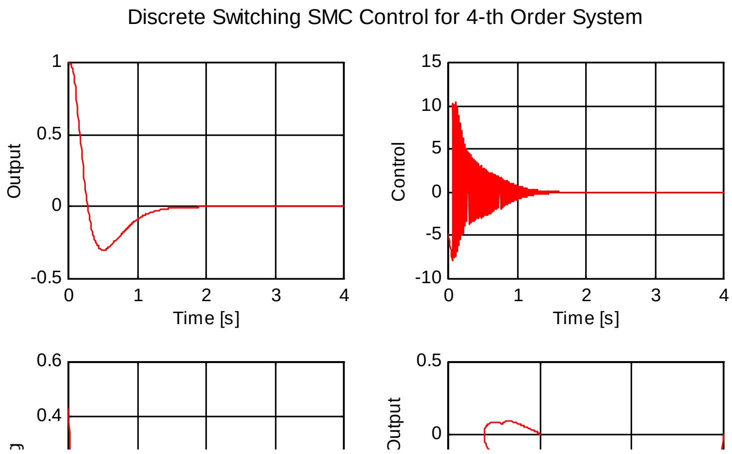


Fig. 5.17: Discrete Switching SMC for 4-th Order System in Example 5.4

- Discrete TanH SMC

Choose $k_s = 20$

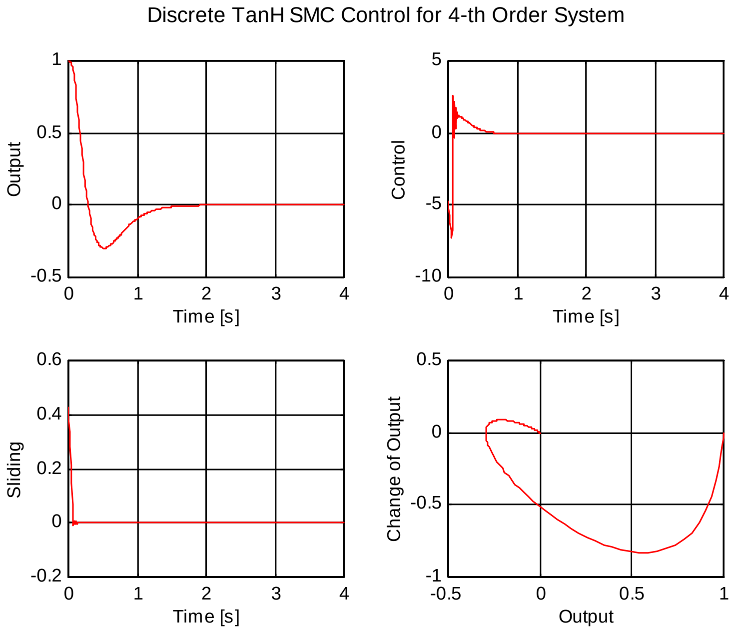


Fig. 5.18: Discrete TanH SMC for 4-th Order System in Example 5.4

5.8.5. Example 5.5: Matched Uncertain System

Consider a matched uncertain linear system in Coleman *et al.* 1994 as in Example 4.2

$$\dot{\mathbf{x}} = \left(\mathbf{A}_c + \Delta\tilde{\mathbf{A}}_c\right).\mathbf{x} + \left(\mathbf{B}_c + \Delta\tilde{\mathbf{B}}_c\right).u$$

where

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2.5 \end{bmatrix}, \quad \Delta\mathbf{A}_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \quad \Delta\mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

Choose $\lambda_H = \begin{bmatrix} -2 & -2 \end{bmatrix}$, then

$$\mathbf{H} = \begin{bmatrix} 0.5254 & 0.5293 & 0.1333 \end{bmatrix}$$

thus

$$\mathbf{K}_e = \begin{bmatrix} 0 & 0.4587 & 0.2293 \end{bmatrix}$$

Choose the sampling time as

$$T_s = 0.01 \text{ sec}$$

since it is about 50 times faster than the desired dynamics (λ_H), then

$$\dot{\mathbf{x}}(k+1) = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right).\mathbf{x}(k) + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right).u(k)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0.0100 & 0 \\ 0 & 1 & 0.0099 \\ 0 & -0.0099 & 0.9753 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.0099 & 0.0049 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.0005 \\ 0.0986 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ -0.0002 \\ -0.0491 \end{bmatrix}$$

- Robust Discrete Linear SMC

  We have

$$\mathbf{K}_p = \begin{bmatrix} 2.6268 & 3.3054 & 0.9960 \end{bmatrix}$$

Choose $\delta = 5$, then Corollary 5.2 yields

$$\mathbf{K}_r = \begin{bmatrix} 2.6268 & 2.6467 & 0.6667 \end{bmatrix}$$

thus

$$\mathbf{K} = \begin{bmatrix} 5.2536 & 6.4107 & 1.8920 \end{bmatrix}$$

Robust Discrete Linear SMC for 3-rd Order Matched Uncertain System

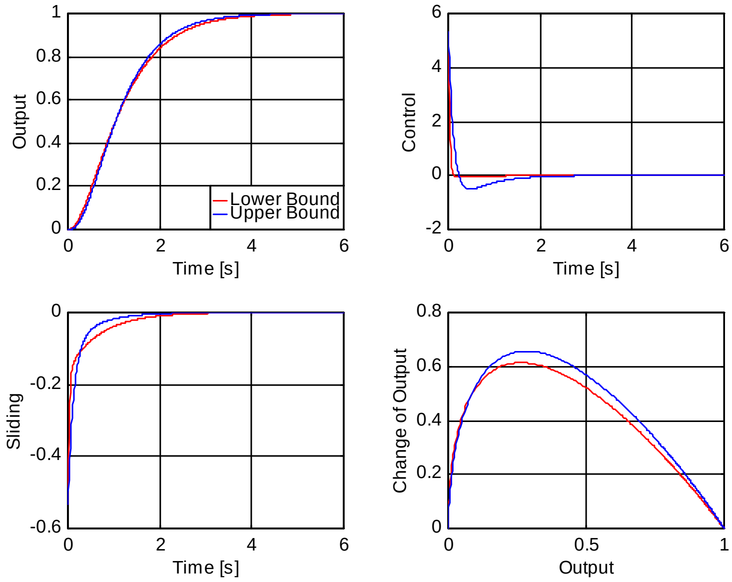


Fig. 5.19: Robust Discrete Linear SMC under Matched Uncertain 3-rd Order System in Example 5.5.

- Robust Discrete Switching SMC

  We have

$$\mathbf{K}_p = \begin{bmatrix} 0 & 0.2 & 0.1 \end{bmatrix}$$

choose $\delta = 5$, then Proposition 5.4 yields

$$\mathbf{K}_r = \begin{bmatrix} 2.6667 & 2.6667 & 0.6667 \end{bmatrix}$$

Robust Discrete Switching SMC Control for Matched Uncertain 3-rd Order System

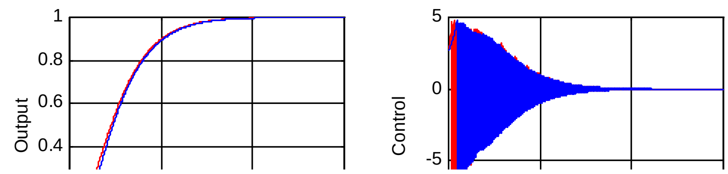


Fig. 5.20: Robust Discrete Switching SMC under Matched Uncertain 3-rd Order System in Example 5.5.

- Robust Discrete TanH SMC

Choose $k_s = 40$

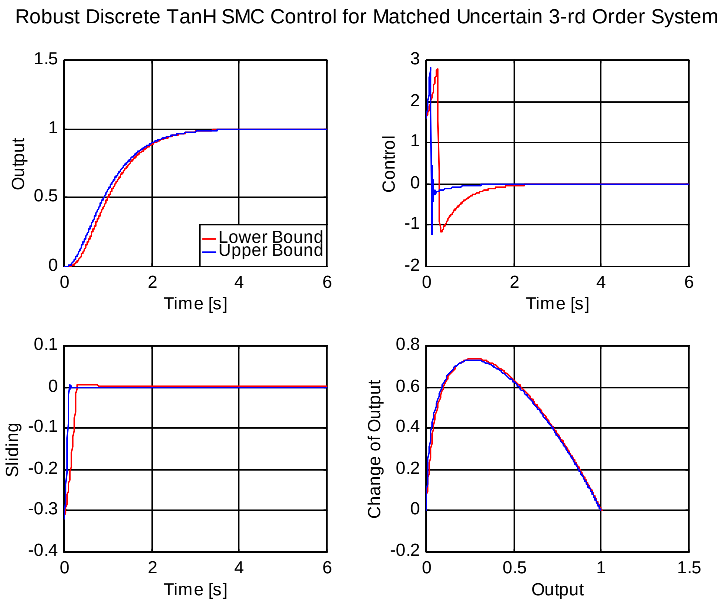


Fig. 5.21: Robust Discrete TanH SMC under Matched Uncertain 3-rd Order System in Example 5.5.

5.8.6. Example 5.6: Un-Matched Uncertain System Under Disturbance

Consider an un-matched uncertain linear system under disturbance in Chen *et al.* 1989 as in Example 4.4

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u + \mathbf{W}v, \quad |v| \le 1$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 1 \\ 8 & -7 & -8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.3 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with the disturbance is chosen as follows for simulation

$$v = \sin(50t)$$

Choose $\lambda_H = [-5 \quad -5]$, then

$$\mathbf{H} = [20.6897 \quad 6.8966 \quad 0.6897]$$

thus

$$\mathbf{K}_e = [41.7241 \quad 15.8621 \quad 1.3793]$$

Choose the sampling time as

$$T_s = 0.01 \text{ sec}$$

since it is about 50 times faster than the desired dynamics (λ_H), then

$$\mathbf{A} = \begin{bmatrix} 1 & 0.01 & 0 \\ 0.0404 & 1 & 0.0096 \\ 0.0755 & -0.0669 & 0.9228 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.0001 \\ 0.0125 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0.0001 & 0 & 0 \\ 0.0202 & 0.0001 & 0 \\ 0.0474 & 0.0002 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.0029 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0.01 \\ 0.0002 \\ 0.0004 \end{bmatrix}$$

- Robust Discrete Linear SMC

We have

$$\mathbf{K}_p = [99.5862 \quad 25.4483 \quad 2.4828]$$

Choose $\delta = 10$ and $\delta_p = 0.2$, then Corollary 5.2 yields

$$\mathbf{K}_r = [206.8966 \quad 68.9655 \quad 6.8966]$$

and $$\mathbf{K}_d = [124.1379 \quad 41.3793 \quad 4.1379]$$

thus $$\mathbf{K} = [472.3448 \quad 151.6552 \quad 14.8966]$$

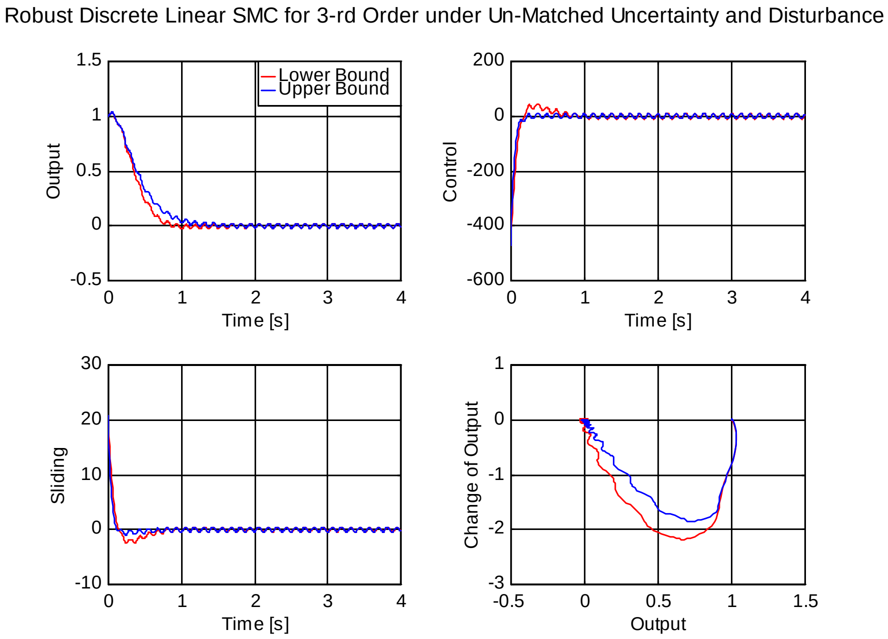


Fig. 5.22: Robust Discrete Linear SMC under Un-Matched Uncertainty and Disturbance in Example 5.6.

Remark 5.6: Temporary Violation of Discrete Sliding Condition

The discrete sliding condition is violated in the region where $|s(k+1)| > |s(k)|$. The proof of Theorem 5.6 has shown that when $|s(k)|$ increases up to a certain magnitude then the condition is satisfied and thus $|s(k)|$ reduces to zero.

- Robust Discrete Switching SMC

We have

$$\mathbf{K}_p = [25.0307 \quad 0.1247 \quad 0.0004]$$

and $$k_o = 29.9396$$

choose $\delta = 2$, then Proposition 5.4 yields

$$\mathbf{K}_r = [45.2387 \quad 15.0796 \quad 1.5079]$$

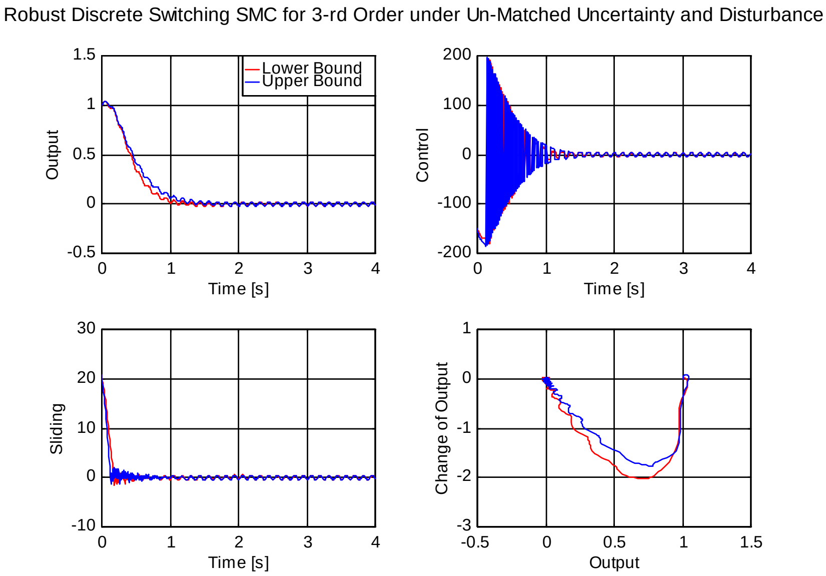


Fig. 5.23: Robust Discrete Switching SMC under Un-Matched Uncertainty and Disturbance in Example 5.6.

- Robust Discrete TanH SMC

Choose $k_s = 2$, then

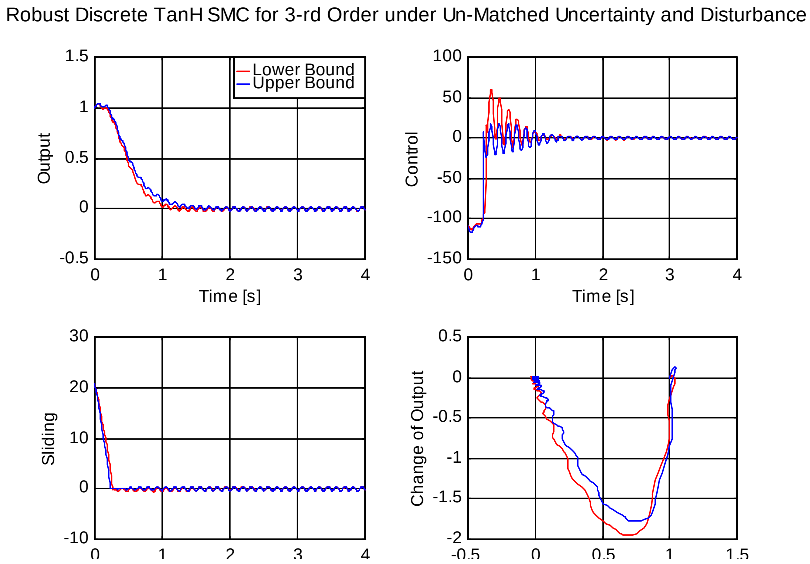


Fig. 5.24: Robust Discrete TanH SMC under Un-Matched Uncertainty and Disturbance in Example 5.6.

5.8.7. Example 5.7: Robust Sliding-Mode Observer under Un-Matched Uncertainty and Disturbance

The controller parameters as in Example 5.6 above, choose the *discrete-time* sliding-mode observer dynamics 2 times faster than the linear SMC dynamics, then Theorem 5.10 yields

$$\mathbf{L} = \begin{bmatrix} 1.6865 \\ 14.1978 \\ -4.0141 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}_o = \mathbf{A} - \mathbf{LC} = \begin{bmatrix} -0.6863 & 0.01 & 0 \\ -14.1574 & 0.9999 & 0.0096 \\ 4.0896 & -0.0669 & 0.9228 \end{bmatrix}$$

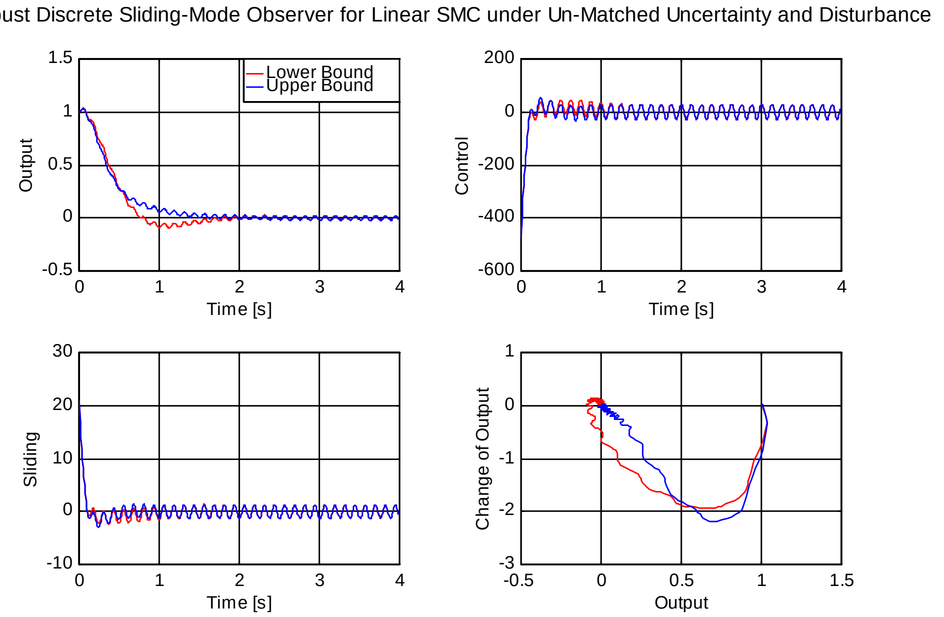


Fig. 5.25: Robust Discrete Sliding-Mode Observer for Linear SMC under Un-Matched Uncertainty and Disturbance in Example 5.7.

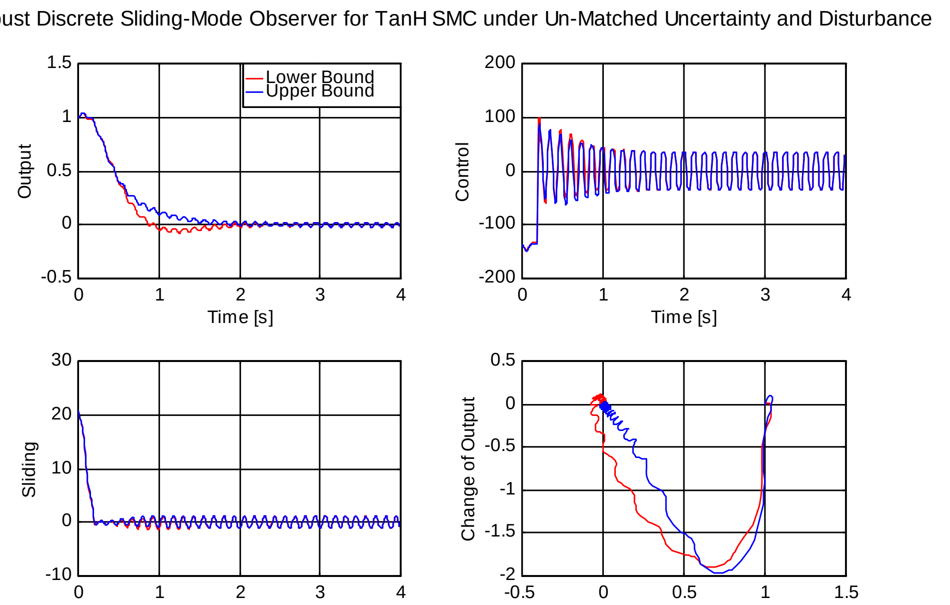


Fig. 5.26: Robust Discrete Sliding-Mode Observer for TanH SMC under Un-Matched Uncertainty and Disturbance in Example 5.7.

5.8.8. Example 5.8: Robust Sliding-Mode Observer under Converted Matched Uncertainty and Disturbance

Example 4.8 is discretized with $T_s = 0.01\,\text{sec}$ to have

$$\mathbf{A} = \begin{bmatrix} 1 & 0.01 & 0 \\ 0.0019 & 1 & 0.0096 \\ 0.3844 & -0.0269 & 0.9230 \end{bmatrix},\quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.0001 \\ 0.0125 \end{bmatrix},\quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0.001 & 0.0001 & 0 \\ 0.2018 & 0.0202 & 0.0001 \end{bmatrix},\quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.0029 \end{bmatrix}$$

By Proposition 5.2, we can use the analog SMC in Example 4.8 as a discrete SMC for this example. However, we have to design a discrete observer by choosing sliding-mode observer dynamics 2 times faster than linear SMC, then Theorem 5.10 yields

$$\mathbf{L}=\begin{bmatrix}1.9330\\14.5241\\3.5990\end{bmatrix},\quad \mathbf{A}_e=\mathbf{A}-\mathbf{LC}=\begin{bmatrix}-0.9330 & 0.01 & 0\\-14.5221 & 1 & 0.0096\\-3.2146 & -0.0269 & 0.9230\end{bmatrix}$$

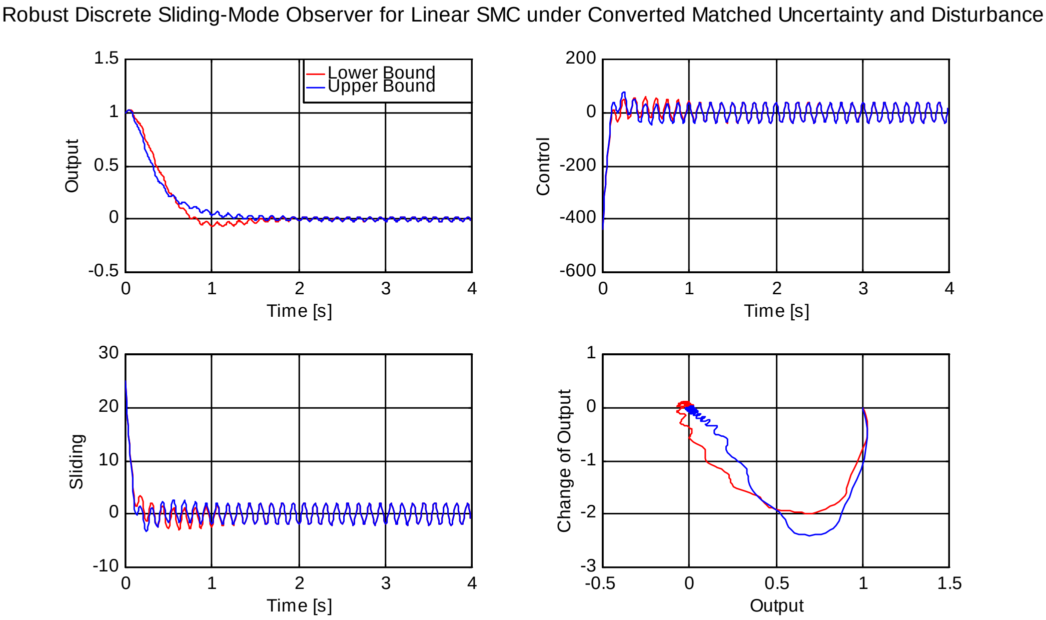


Fig. 5.27: Robust Discrete Sliding-Mode Observer for Linear SMC under Converted Matched Uncertainty and Disturbance in Example 5.8.

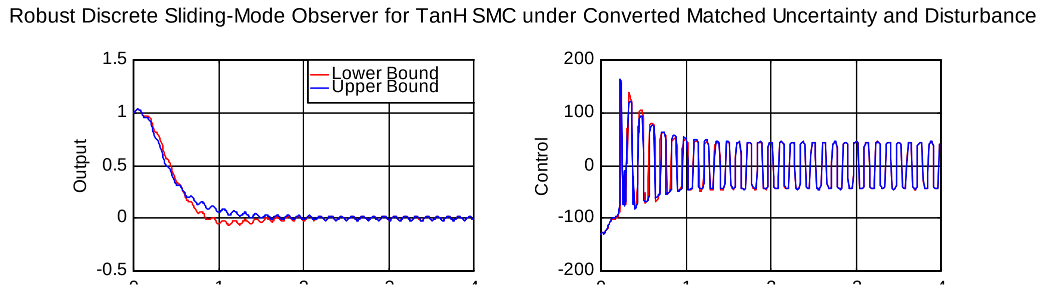


Fig. 5.28: Robust Discrete Sliding-Mode Observer for TanH SMC under Converted Matched Uncertainty and Disturbance in Example 5.8.

5.9. CONCLUSION

In both designs from Utkin and Furuta, there is a boundary layer corresponding to the term $|s_k| < \beta$, so there is no chattering problem, but strictly speaking it is pseudo-sliding mode control while it is not the case for the new linear discrete sliding mode control. In the proposed design in Section 2, there is no condition on β as in the Furuta's approach (Furuta 1990, Pieper *et al* 1992). The main advantages of the proposed discrete linear SMC are that the control function is simple and the performance is predictable (system-eigenvalues). Furthermore, it is applicable to both matched and unmatched uncertainty. There is a condition on the sampling rate for uncertain dynamical systems but not for deterministic systems. However, this condition can be relaxed by increasing the sampling rate as shown in Example 2 above.

A convergence condition has been proposed in Sarpturk *et al.* 1987 to guarantee system stability while the hyperplane has not been based on hyperplane-eigenvalues. Unfortunately, our simulation of the violation has still produced stable response! In Spurgeon 1992, the convergence condition has been also presented, however there has been no development to support this condition. The hyperplane design has been based on hyperplane-eigenvalues as in Utkin *et al.* 1978.

A new robust discrete-time linear SMC design has been fully developed. Its analysis has been used to explore the true nature of SMC including the invariance property and the stability problem. In the *New Robust Discrete-Time Linear SMC Design*, the control function is partitioned into 3 components: equivalent control, reaching control, and perturbation control. The control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem in the standard SMC design. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework. In the current SMC literature, a discrete-time SMC has been presented in Sarpturk *et al.* 1987, Sira-Ramirez 1991 and Spurgeon 1992, but a robust discrete-time SMC has not been fully developed.

Chapter 5

A New Robust Discrete-Time Sliding Mode Controller-Observer Design

Chapter 6

Robust Sliding-Mode Fuzzy Controller Design

6.1. INTRODUCTION

Since the invention of the first fuzzy controller by Mamdani in 1974, fuzzy controllers have been found successfully in numerous industrial applications such as cement-kiln process control, automatic train operation, camcorder autofocusing, crane control, etc. These systems could be classified as slow systems. We find the best fuzzy structure (membership function, fuzzification, defuzification, rulebase, fuzzy inference) applicable to both slow and fast systems

In a fuzzy control, a typical fuzzy rulebase with 2 entries is used for an 2-nd order system. This approach can be extended using a mapping for a 3-rd order system, however it may not be convenient for a higher order system unless it can be decomposed into some 2-nd order sub-systems. In a sliding-mode fuzzy control, sliding variable and its change are in place of error and its change where all system states can be included in the sliding variable (Hwang *et al.* 1992).

The main concern in control engineering is the stability problem. In the current fuzzy control literature, the stability of a fuzzy control is based on a fuzzy model that is inferred from mathematical models (Tanaka *et al.* 1992, Ishigame *et al.* 1993). The stability problem of the sliding-mode fuzzy control in Hwang *et al.* 1992 has been unsolved.

In this chapter, we prove that a typical fuzzy rulebase can satisfy the Lyapunov sliding condition so the stability is guaranteed by the Lyapunov stability theorem. On this basis, to design a stable sliding-mode fuzzy controller, a fuzzy mechanism is used to minimize a sliding variable s instead of using the sliding condition as in the sliding mode control, so we can obtain the invariance property of the sliding mode. In a typical fuzzy rulebase, it may not be convenient to use more than 2 entries, we can use 1 entry for s and the other for sum of s, and hence a possible steady-state error may be eliminated by this I-action.

In a fuzzy control, the problems are how to choose the gains for error and its change; and a possible chattering (limit cycle). In the current fuzzy control literature, these gains are chosen by trial and error or chosen unity without justification, and the unit circle is used to analyze the chattering, not to solve this problem. Using the sliding-mode control theory, these gains can be determined by a hyperplane and the chattering problem can be solved since the system dynamics are included through these gains (Example 6.1 & 6.2).

On the basis of the fuzzy identification in Tanaka *et al.* 1992 and Ishigame *et al.* 1993, we develop a new fuzzy identification scheme which is simpler and more practical. The fuzzy inference will be used to obtain the most potential model from some rough mathematical models from experiments using a proposed practical system identification. Due to the robustness, a rough system model is required rather than an elaborate mathematical model as in a conventional control, a practical system identification is presented for this purpose. A fuzzy model by the proposed scheme can be a solution to the conservative problem.

6.2. FUZZY CONTROL

The following is a typical fuzzy-logic control system.

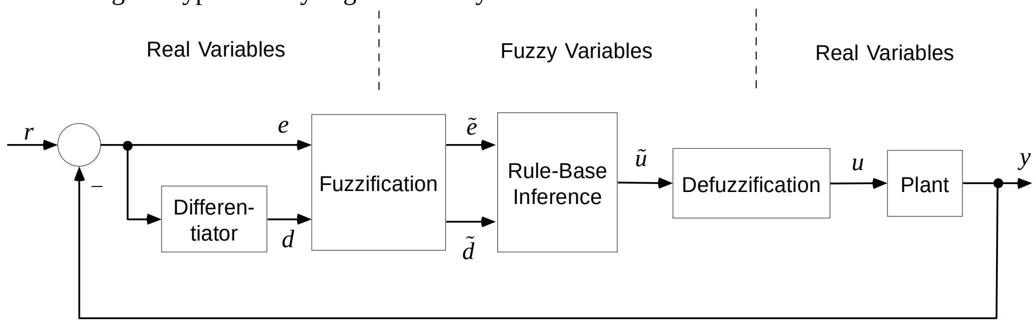


Fig. 6.1: Fuzzy Control Architecture

where

- $r, e, d, u,$ and y are real variables
- $\tilde{e}, \tilde{d},$ and $\tilde{u}$ are fuzzy variables
- fuzzification transforms real input variables into fuzzy variables
- rule-base inference is used to compute a fuzzy control variable
- defuzzification converts the fuzzy control variable into a real variable to control the system

6.2.1. Membership Functions

Membership function can be a triangle or bell function.

6.2.1.1. Triangle Membership Function

Triangle membership function is defined as

$$\mu_i(x, a_i, b_i, c_i) = \max\left(0, \frac{x - a_i}{b_i - a_i}\right) + \min\left(0, \frac{-(c_i - a_i)(x - b_i)}{(b_i - a_i)(c_i - b_i)}\right) + \max\left(0, \frac{x - c_i}{c_i - b_i}\right) \tag{6.1}$$

where a_i, b_i, c_i are defined as in the following figure

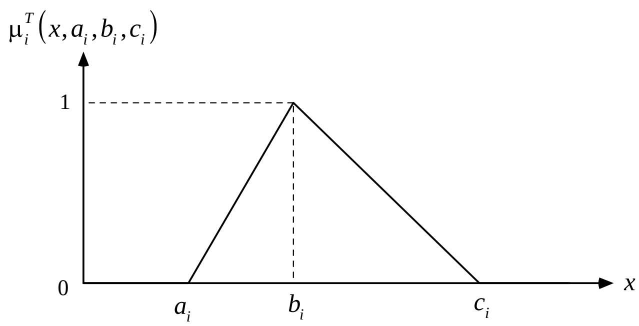


Fig. 6.2: Triangle Membership Function

6.2.1.2. Bell Membership Function (Gaussian Distribution Function)

Gaussian membership function (Bell function) is defined as

$$\mu_i(x, a_i, b_i) = \exp\left\{-\left(\frac{x - a_i}{b_i}\right)^2\right\} \tag{6.2}$$

where a_i, b_i are defined as in the figure below

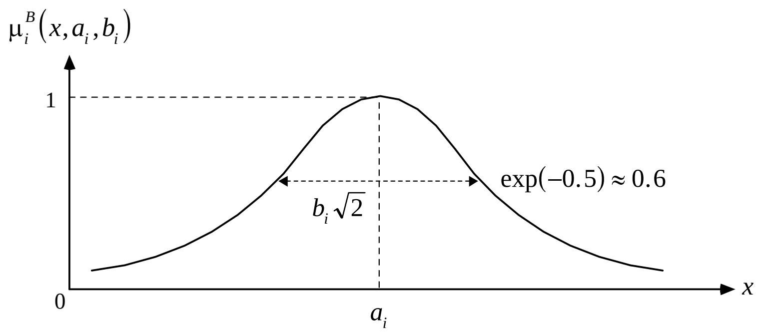


Fig. 6.3: Bell Membership Function

6.2.2. Fuzzification

The followings are some typical fuzzifications

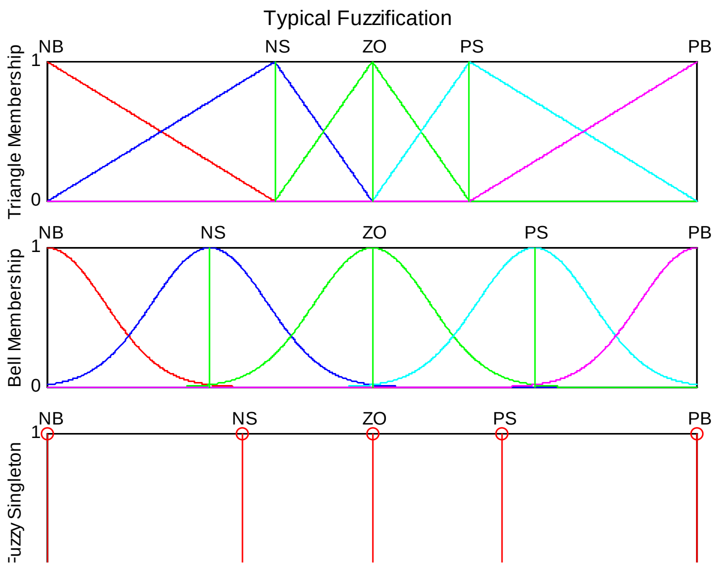


Fig. 6.4: Typical Fuzzifications

Remark 6.1: Normal and Weighted Fuzzifications

The fuzzification of linear distribution is termed as a *normal* fuzzification due to its uniform distribution. The fuzzification of nonlinear distribution is termed as a *weighted* fuzzification since the distribution is weighted in the middle.

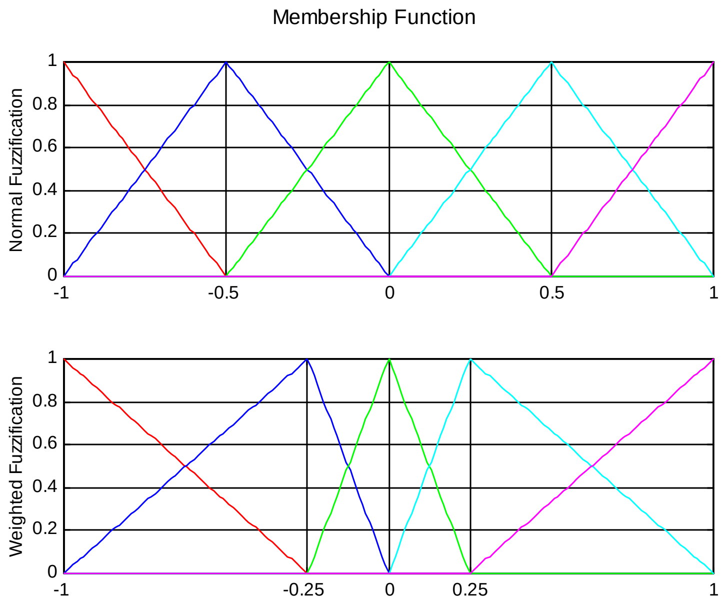


Fig. 6.5: Normal and Weighted Fuzzifications

6.2.3. Fuzzy Rulebase

A fuzzy rulebase can take 3 forms: soft, sharp and full rulebase.

6.2.3.1. Soft Rulebase

A soft rulebase has a *little output change* around the middle. All input and output variables have the same number of values.

d **e**	$\mathbf{NB}_\mathbf{d}$	$\mathbf{NS}_\mathbf{d}$	$\mathbf{ZO}_\mathbf{d}$	$\mathbf{PS}_\mathbf{e}$	$\mathbf{PB}_\mathbf{d}$
$\mathbf{NB}_\mathbf{e}$	PB_u	PB_u	$\mathbf{PS}_u$	$\mathbf{PS}_u$	ZO_u
$\mathbf{NS}_\mathbf{e}$	PB_u	$\mathbf{PS}_u$	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$
$\mathbf{ZO}_\mathbf{e}$	$\mathbf{PS}_u$	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$	$\mathbf{NS}_u$
$\mathbf{ZO}_\mathbf{d}$	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$	$\mathbf{NS}_u$	NB_u
$\mathbf{PB}_\mathbf{e}$	ZO_u	$\mathbf{NS}_u$	$\mathbf{NS}_u$	NB_u	NB_u

Table 6.1: Soft Fuzzy Rulebase

6.2.3.2. Sharp Rulebase

A sharp rulebase has a *remarkable output change* around the middle. All input and output variables have the same rule number.

d / e	NB_d	NS_d	ZO_d	PS_e	PB_d
NB_e	PB_u	PB_u	PB_u	$\mathbf{PS}_u$	ZO_u
NS_e	PB_u	PB_u	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$
ZO_e	PB_u	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$	NB_u
ZO_d	$\mathbf{PS}_u$	ZO_u	$\mathbf{NS}_u$	NB_u	NB_u
PB_e	ZO_u	$\mathbf{NS}_u$	NB_u	NB_u	NB_u

Table 6.2: Sharp Fuzzy Rulebase

6.2.3.3. Full Rulebase

Every output variable has its own value in a full rulebase. If input variable has *n* rules, an output variable has $2n-1$ rules.

d / e	NB_d	NS_d	ZO_d	PS_e	PB_d
NB_e	PB_u	PM_u	PS_u	$\mathbf{PZ}_u$	ZO_u
NS_e	PM_u	PS_u	$\mathbf{PZ}_u$	ZO_u	$\mathbf{NZ}_u$
ZO_e	PS_u	$\mathbf{PZ}_u$	ZO_u	$\mathbf{NZ}_u$	NS_u
ZO_d	$\mathbf{PZ}_u$	ZO_u	$\mathbf{NZ}_u$	NS_u	NM_u
PB_e	ZO_u	$\mathbf{NZ}_u$	NS_u	NM_u	NB_u

Table 6.3: Full Fuzzy Rulebase

where

NB, NM, NS, NZ, ZO, PZ, PS, PM, PB: negative big, negative medium, negative small, negative zero, zero, positive zero, positive small, positive medium, positive big.

Remark 6.2: Output Change in Full Rulebase

The full rulebase also has *sharp control change* around the middle.

6.2.4. Fuzzy Inference

Fuzzy inference can be the *minimum* or *product* method.

Minimum fuzzy inference is defined as

$$\mu^U(e,d) = \min\left\{\mu^E(e), \mu^D(d)\right\} \tag{6.3}$$

Product fuzzy inference is defined as

$$\mu^U(e,d) = \mu^E(e) \times \mu^D(d) \tag{6.4}$$

6.2.5. Defuzzification

Defuzzification can be the mean-of-maxima or centroid method:

Mean-of-Maxima Defuzzification is defined as

$$u(e,d) = \frac{1}{m}\sum_{i=1}^{m} u_i, \qquad \mu_U(u_i) = \max\{\mu_U(u_1), \mu_U(u_2), \cdots\} \tag{6.5}$$

Centroid defuzzification is defined as

$$u(e,d) = \frac{\mu_U(u_1)\,u_1 + \mu_U(u_2)\,u_2}{\mu_U(u_1) + \mu_U(u_2)} \tag{6.6}$$

6.3. STABILITY OF FUZZY CONTROL

We will analyze the stability of a conventional fuzzy control for a second order system, since it is most suitable for these systems.

Lemma 6.1: Stability of Sliding Condition

Consider a hyperplane of a second order system

$$s(t) = [\lambda, \quad 1].\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} = \lambda.e(t) + \dot{e}(t), \qquad \lambda > 0 \tag{6.7}$$

if the following sliding condition is satisfied

$$s(t).\dot{s}(t) < 0 \tag{6.8}$$

then $e(t)$ decays to *zero* with a time-constant of λ .

Proof:

Consider a Lyapunov function

$$V = \tfrac{1}{2}s^2(t) > 0$$

then by Eq.(6.8), we have

$$\dot{V} = s(t).\dot{s}(t) < 0$$

so *V*, and hence *s*, decreases to *zero* since $V > 0$. Thus

$$\lambda.e(t) + \dot{e}(t) = 0 \quad \Rightarrow \quad e(t) = e(0).\exp(-\lambda.t) \to 0 \tag{6.9}$$

since $\lambda > 0$.

Q.E.D

Assumption 6.1: Bounds of Uncertain Nonlinear Systems

In the following nonlinear second order system

$$\ddot{e} = f(e,\dot{e}) + g(e,\dot{e}).u \tag{6.10}$$

where

$$|f(e,\dot{e})| \le \phi(e,\dot{e}), \qquad |g(e,\dot{e})| \ge \gamma(e,\dot{e})$$

the polarity of $g(e,\dot{e})$ is unchanged and without loss of generality, assume that $g(e,\dot{e}) > 0$. If $g(e,\dot{e}) < 0$, then let $v = -u$ and *v* will be determined with $g(e,\dot{e}) > 0$ then $u = -v$.

Since there is no *Z*-transform for a nonlinear system, we will consider in continuous time. In view of Table 6.1, we can draw a design law that the larger amplitude of antecedents e and $\dot{e}$, the larger amplitude of the consequent, so we need to determine the bound of the consequent u in Table 6.1 for the system in Eq.(6.30) to be stable. In doing so, we propose the following theorem

Theorem 6.1: Stability Criterion for Fuzzy Control

Consider a nonlinear second order system

$$\ddot{e} = f(e,\dot{e}) + g(e,\dot{e}).u$$

where

$$|f(e,\dot{e})| \le \phi(e,\dot{e}), \qquad |g(e,\dot{e})| \ge \gamma(e,\dot{e})$$

under Assumption 6.1, if u as the consequent in Table 6.1 has the following bound

$$\sup|u| = \frac{\phi(e_{\max},\dot{e}_{\max}) + |\dot{e}_{\max}|}{\gamma(e_{\max},\dot{e}_{\max})} \tag{6.11}$$

then the system is asymptotically stable. It is more feasible to stabilize a slower dynamic system since this bound is lower for smaller $|\dot{e}_{\max}|$.

Proof:

Let

$$s = e + \dot{e}$$

then

$$\dot{s} = \dot{e} + \ddot{e} = \dot{e} + f(e,\dot{e}) + g(e,\dot{e}).u \tag{6.12}$$

The ZO-line divides Table 6.1 into 2 areas, $s > 0$ in the upper and $s < 0$ in the lower. Since Table 6.1 is symmetric across the ZO-line, it is sufficient to prove for the case $s > 0$, then by Table 6.1 we have $u < 0$. For the most conservative case, except ZO, we can choose all as NB and PB with the magnitude of the bound in Eq.(6.11). To satisfy Table 6.1, u can be found from

$$u = -\frac{\phi(e,\dot{e}) + |\dot{e}|}{\gamma(e,\dot{e})}$$

substitute into Eq.(6.12), we obtain

$$\dot{s} = \dot{e} + f(e,\dot{e}) - g(e,\dot{e}).\frac{\phi(e,\dot{e}) + |\dot{e}|}{\gamma(e,\dot{e})} < \dot{e} + f(e,\dot{e}) - \left[\phi(e,\dot{e}) + |\dot{e}|\right] < 0$$

the by Lemma 6.1, $e(t)$ decays to *zero* with a time constant of 1.

If there exists a layer in the vicinity of ZO-line within which the value of u is smaller than the required boundary value; then the system states are bounded within this layer because once they exit this layer, u satisfies the required bound and the states are pulled back into this layer. This phenomenon is the chattering problem in a sliding mode control and is the chattering in a fuzzy control.

Q.E.D

We have the following proposition as a guide to choose a sampling time.

Proposition 6.1: Selection of Sampling Time

In fact, the consequent in Table 6.1 is determined in such a way that it reduces the antecedents to *zero*. A fuzzy controller is usually implemented in discrete form, the sampling time is chosen fast enough accordingly to the system dynamics. Otherwise the consequent will over-reduces and a positive e goes to negative and vice versa, instead of decaying to *zero*. Similarly, the sampling time in Theorem 1 is chosen fast enough accordingly to the system dynamics, otherwise the condition $s\dot{s} < 0$ over-reduces s then the next sample of s will cross the ZO-line and a boundary layer will be created. In a discrete-time sliding-mode control, there is also a condition on the sampling time.

6.4. Case Studies of Fuzzy Control Structure

Since the invention of the first fuzzy controller by Mamdani in 1974, fuzzy controllers have been found successfully in numerous industrial applications such as cement-kiln process control, automatic train operation, camcorder autofocusing, crane control, etc. These systems could be classified as slow systems.

In this section, we will consider applications of fuzzy controllers in a slow and fast systems under different rule numbers, under different fuzzifications (normal and weight) and under different rulebases (soft, sharp and full). The product method and the centroid defuzzification are used to compute a control output. The gains of error and its change are chosen as unity.

Second-order systems are used since the following reasons:

- Most of control actuators are servo motor of second order and a system can be decomposed into some 2-order subsystems (Remark 6.11)
- It may be the best application of a fuzzy controller with error and its change in 2-order systems since it can use full system states in controlling

Remark 6.3: Fuzzy Control Variable

Control output is fuzzified within the range $\left[-U_{rng}, U_{rng}\right]$ where U_{rng} is chosen to satisfy the constraint of a maximum control effort.

6.4.1. Fuzzy Control for Slow System

Consider the slow servo motor system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{A} = \begin{bmatrix} 0, & 1 \\ 0, & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

6.4.1.1. Soft Rulebase

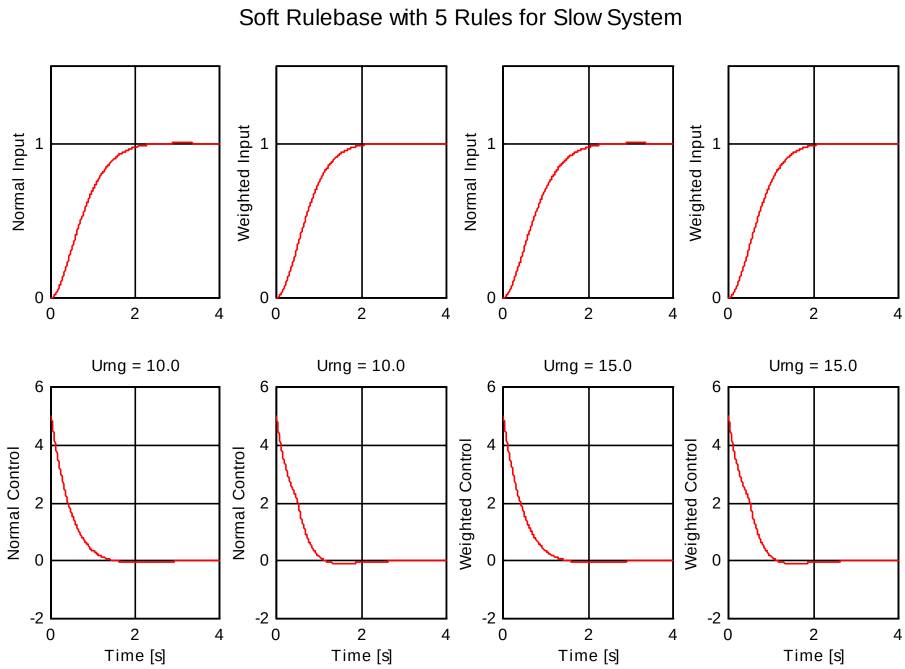


Fig. 6.6: Fuzzy Control of **5 Rules** under Soft Rulebase with Different Fuzzifications for Slow System

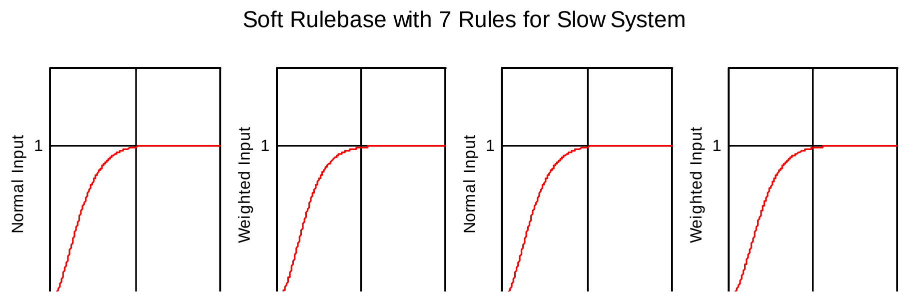


Fig. 6.7: Fuzzy Control of **7 Rules** under Soft Rulebase with Different Fuzzifications for Slow System

6.4.1.2. Sharp Rulebase

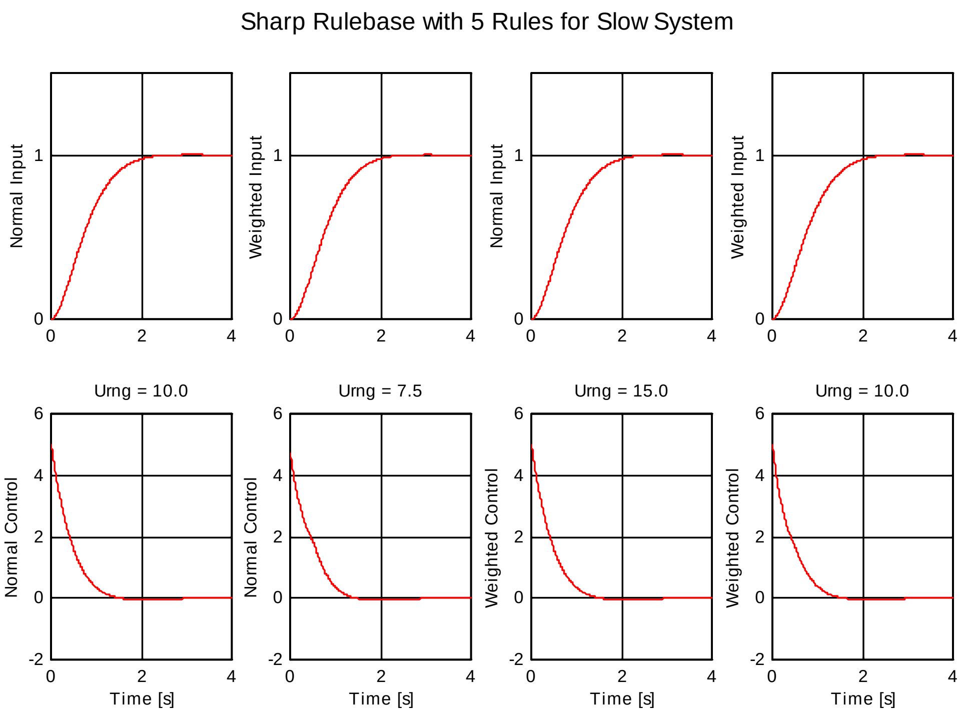


Fig. 6.8: Fuzzy Control of **5 Rules** under Sharp Rulebase with Different Fuzzifications for Slow System

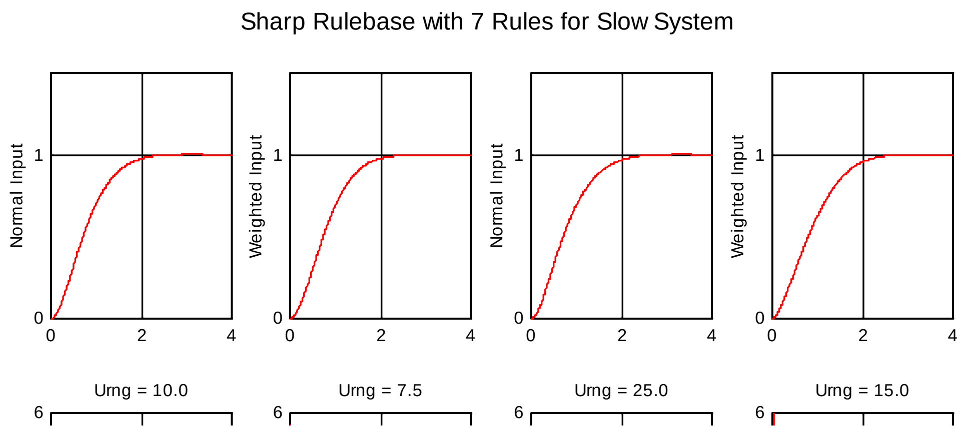


Fig. 6.9: Fuzzy Control of **7 Rules** under Sharp Rulebase with Different Fuzzifications for Slow System

6.4.1.3. Full Rulebase

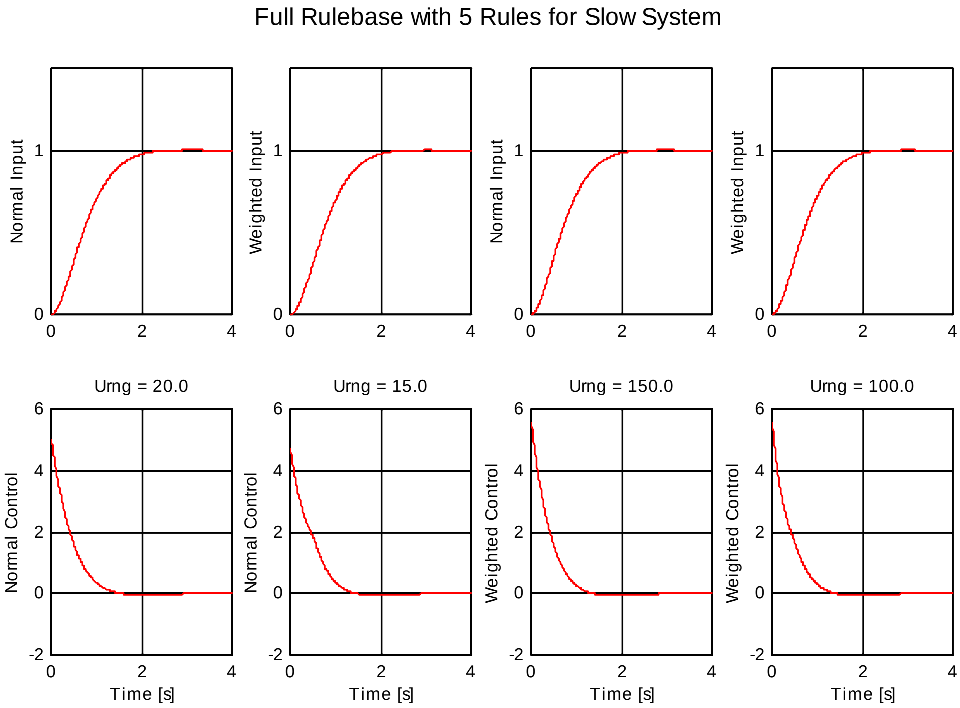


Fig. 6.10: Fuzzy Control of **5 Rules** under Full Rulebase with Different Fuzzifications for Slow System

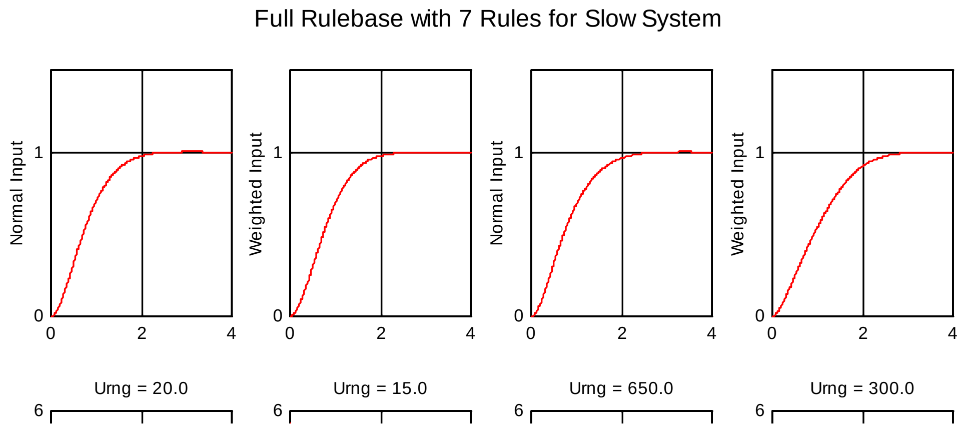


Fig. 6.11: Fuzzy Control of **7 Rules** under Full Rulebase with Different Fuzzifications for Slow System

6.4.2. Fuzzy Control for Fast System

Consider the fast servo motor system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{A} = \begin{bmatrix} 0, & 1 \\ 0, & -20 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 600 \end{bmatrix}$$

6.4.1.1. Soft Rulebase

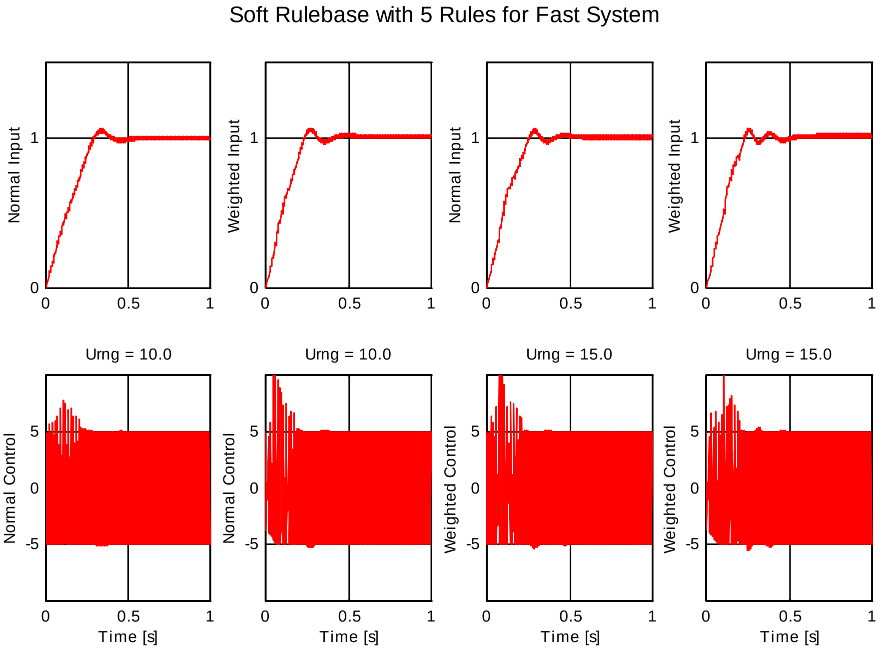


Fig. 6.12: Fuzzy Control of **5 Rules** under Soft Rulebase with Different Fuzzifications for Fast System

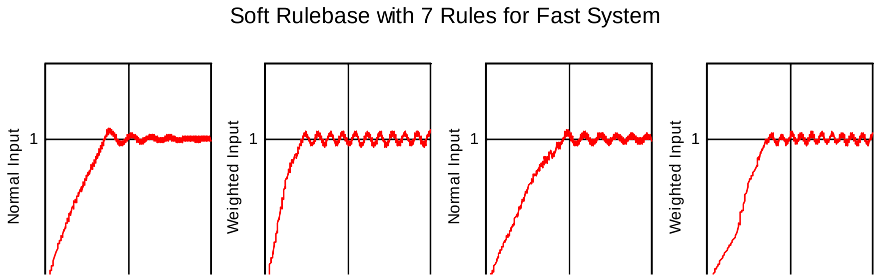


Fig. 6.13: Fuzzy Control of **7 Rules** under Soft Rulebase with Different Fuzzifications for Fast System

6.4.1.2. Sharp Rulebase

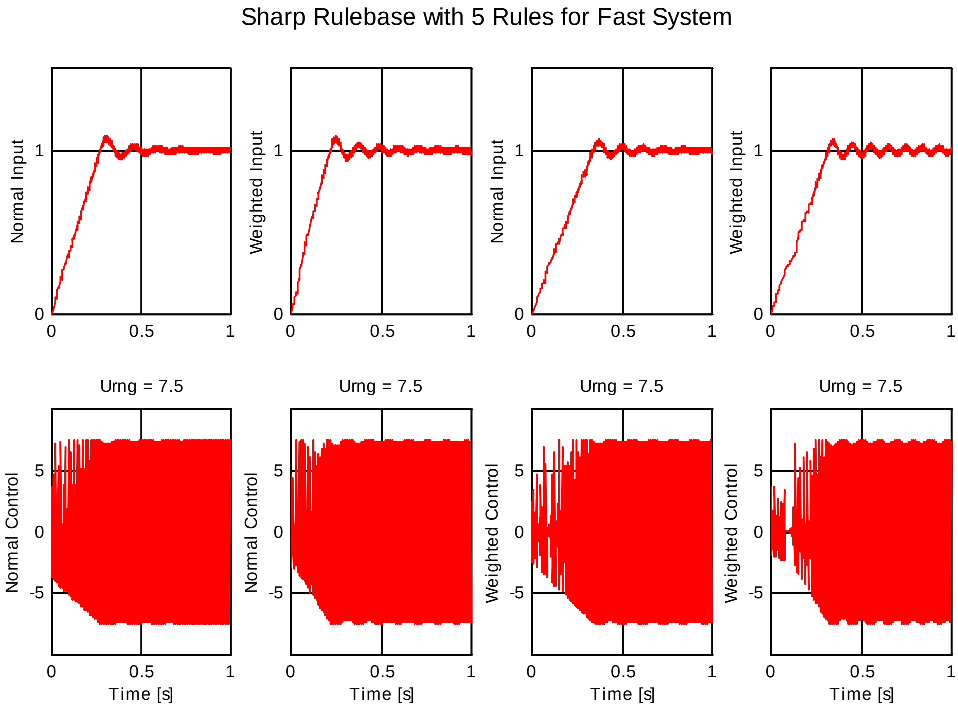


Fig. 6.14: Fuzzy Control of **5 Rules** under Sharp Rulebase with Different Fuzzifications for Fast System

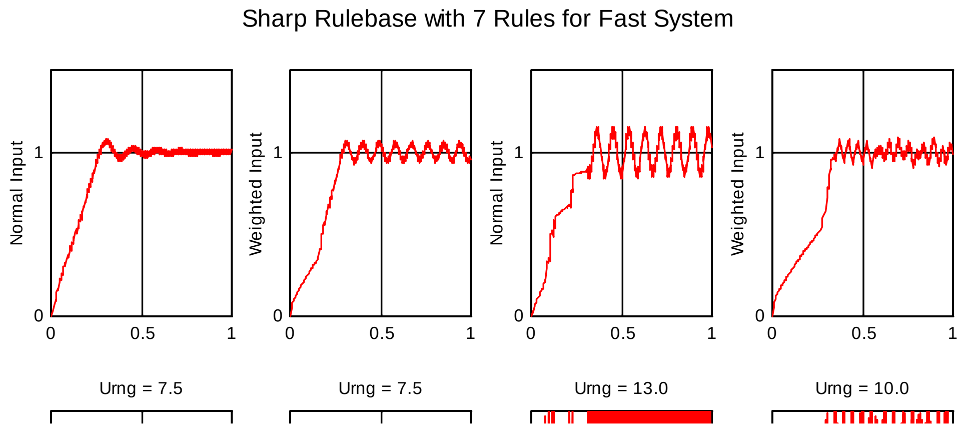


Fig. 6.15: Fuzzy Control of **7 Rules** under Sharp Rulebase with Different Fuzzifications for Fast System

6.4.1.3. Full Rulebase

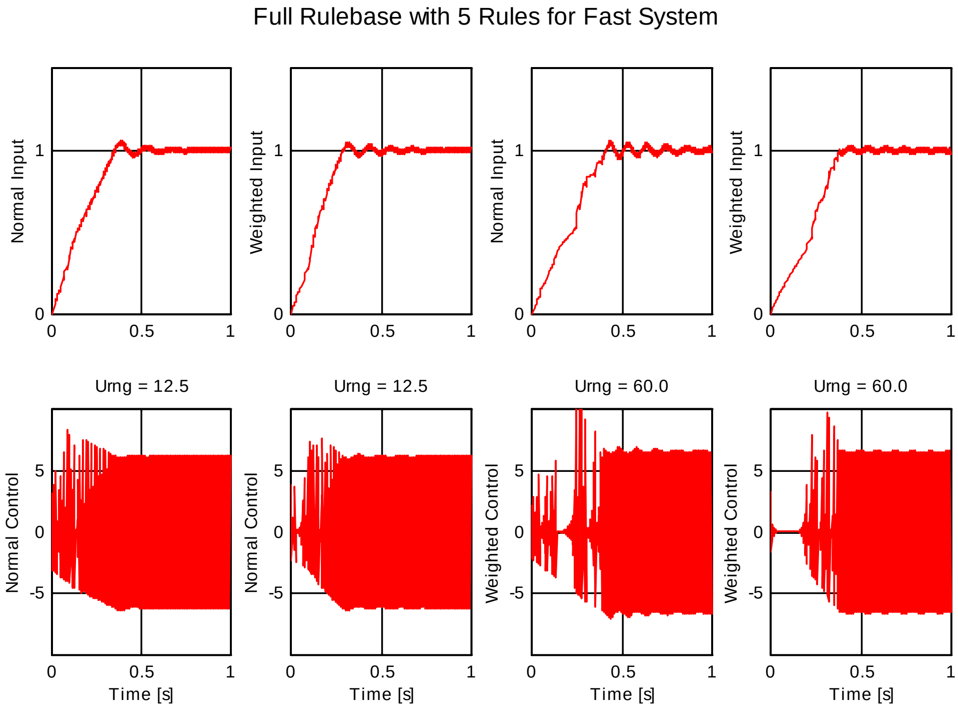


Fig. 6.16: Fuzzy Control of **5 Rules** under Full Rulebase with Different Fuzzifications for Fast System

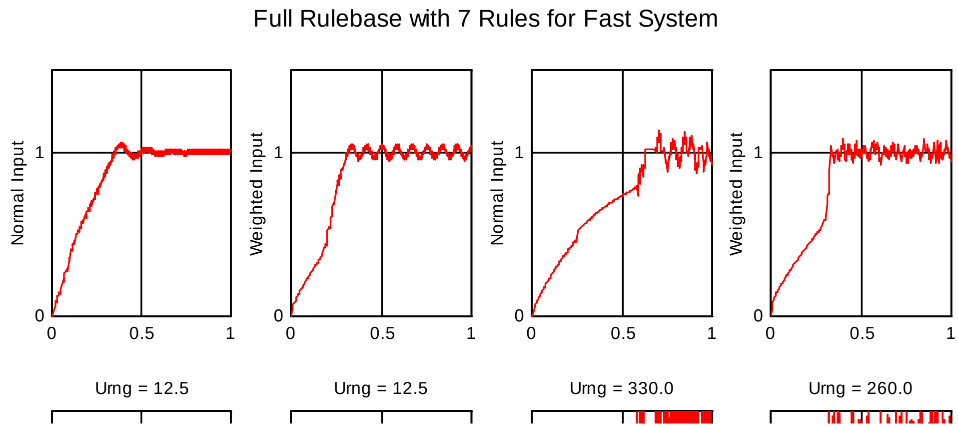


Fig. 6.17: Fuzzy Control of **7 Rules** under Full Rulebase with Different Fuzzifications for Fast System

6.4.3. Discussion

The following remark is proposed some criteria to benchmark applications of fuzzy control for the slow and fast systems

Remark 6.4: Benchmark Criteria

- Failure is defined as the defuzzification range of control is too big than the actual control. For example, in Fig. 6.11, the control range is 650 while the control peak is only 5.

- Success is defined as the defuzzification range of control is less than double inclusively of the actual control. For example, in Fig. 6.6, successes occur if the control range is less than 10 inclusively as the control peak is about 5.

For the slow system, we have

SLOW SYSTEM	Normal Output		Weighted Output	
Soft Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	**10**	**10**	15	15
Urng (7 Rules)	15	12.5	***45***	***30***
Sharp Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	**10**	**7.5**	15	**10**
Urng (7 Rules)	**10**	**7.5**	***25***	***15***
Full Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	20	15	***150***	***100***
Urng (7 Rules)	20	15	***650***	***300***

Remark 6.5: Limitation of Bell Function

If bell function is used instead of triangle function, we have

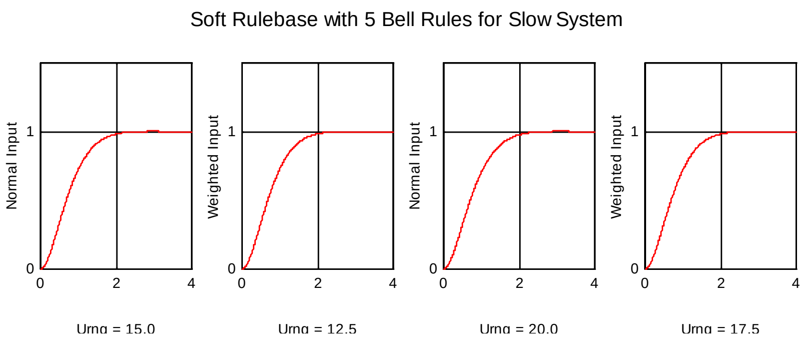


Fig. 6.18: Fuzzy Control of **5 Bell Rules** under Soft Rulebase with Different Fuzzifications for Slow System

The bell function goes to zero faster than the triangle function, so the former requires a larger control range, this is a limitation due to the criteria in Remark 6.4.

Remark 6.6: Centroid Defuzzification more Superior than Mean-of-Maxima Method

If the mean-of-maxima defuzzification is used instead of centroid method, we have the following figure show that the steady-state error is a serious problem, this is consistent with the work in Brae *et al.* 1978, Tong 1978, Larkin 1985, Sharf *et al.* 1985. This problem can be solved using sum of error instead of change of error, *ie.* including I-action. However the control performance may not be satisfactory when the D-action is not included to deal with fluctuations, if it is the case.

Soft Rulebase with 5 MOM Rules for Slow System

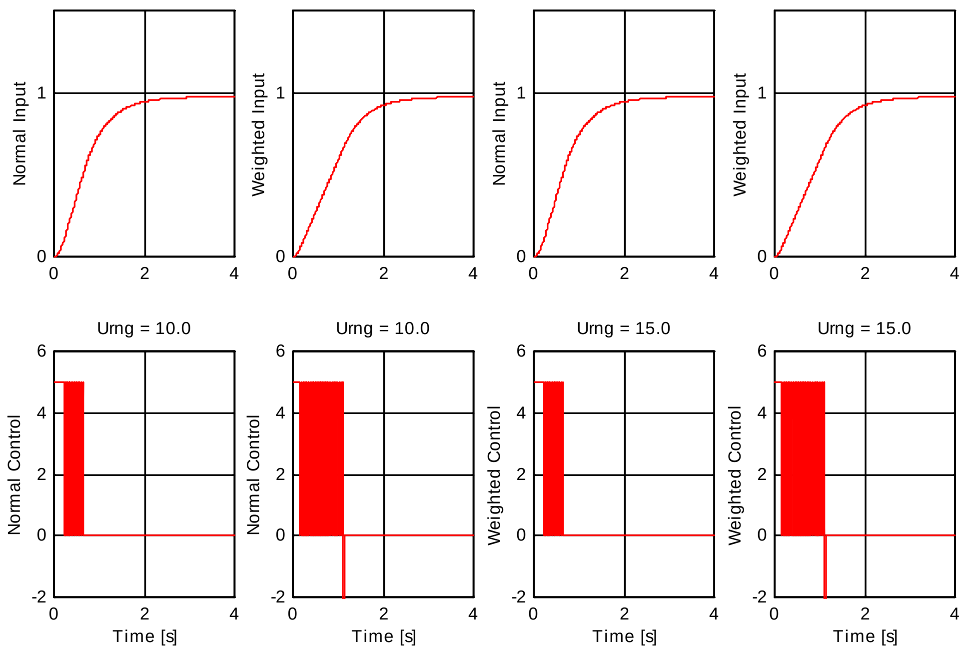


Fig. 6.19: Fuzzy Control of **5 Rules** under Soft Rulebase with Different Fuzzifications for Slow System using Mean-of-Maxima Defuzzification

For the fast system, we have

FAST SYSTEM	**Normal Output**		**Weighted Output**	
Soft Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	**10**	**10**	15	15
Urng (7 Rules)	**10 (Osc.)**	**10 (Osc.)**	**22.5 (Osc.)**	**20 (Osc.)**
Sharp Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	**7.5 (Osc.)**	**7.5 (Osc.)**	**7.5 (Osc.)**	**7.5 (Osc.)**
Urng (7 Rules)	**7.5**	**7.5**	13	10
Full Rulebase	*Normal Input*	*Weighted Input*	*Normal Input*	*Weighted Input*
Urng (5 Rules)	**12.5 (Osc.)**	**12.5 (Osc.)**	***60* (Osc.)**	***60* (Osc.)**
Urng (7 Rules)	**12.5 (Osc.)**	**12.5 (Osc.)**	***330***	***260***

Remark 6.7: Performance of Fuzzy Control for Fast System

5-rule soft rulebase and 7-rule sharp rulebase yield comparable results. However, 7-rule soft rulebase produces less oscillations than 5-rule sharp rulebase. In addition, less rule number requires less computation. 5-rule soft rulebase is thus considered to be better.

Remark 6.8: Weighted Fuzzifications

- Weighted input fuzzification yields superior results due to the weighting around the desired set point
- Weighted output fuzzification produces unsatisfactory results. The control range must be increased for the weighted control range to cover the operating control range, this is a limitation due to the criteria in Remark 6.4

By the criteria in Remarks 6.4 to 6.8, we have the following proposition on successful applications of fuzzy control for both slow and fast systems:

Proposition 6.2: Fuzzy Control Structure

For the best application in slow and fast systems, a fuzzy control can be take the following structure

- triangle membership function
- normal or weighted input and normal output fuzzifications
- soft rulebase with 5 rules
- product inference and centroid defuzzification

Remark 6.9: Limitation of Expensive Computation Effort of Fuzzy Control

A fuzzy computation may be so expensive that it cannot be completed within one sampling time. The maximum sampling time is limited by the dominant system time-constant.

6.5. SLIDING-MODE FUZZY CONTROL

In a conventional fuzzy control, a typical fuzzy rulebase with 2 entries is used for an 2-nd order system, one entry is for an error and the other for change of error. If a steady-state exists, a sum of error may be used instead of change of error to eliminate the steady-state error, then there is no D-action to tackle the fluctuations which may exist in the system dynamics (Example 6.2). This approach can be extended using a mapping for a 3-rd order system, however it may not be convenient for a higher order system. A sliding-mode fuzzy control will use s for the first entry and since the D-action is included in s, the other entry is used for the sum of s to introduce an I-action in elimination of a possible steady-state error. In addition, a sliding-mode fuzzy control is applicable for higher order systems since the system states are included in s.

To control a system effectively, some system information must be used to obtain a model for designing a controller. A neu-net control is based on the numerical data from the system input-output to get a neu-net model, a fuzzy control is based on the system knowledge-base to have a fuzzy model. For a sliding-mode fuzzy control, a hyperplane is designed based on a fuzzy model which is inferred from rough mathematical models using step responses due to the robustness of the sliding mode.

From Theorem 6.1, we have the following corollary as a basis for a sliding-mode fuzzy control

Corollary 6.1: Sliding Variable

In Table 6.1, if

$$s = e + \dot{e}$$

is used as a single entry, then we have Table 6.4 of single entry to determine the consequent u to obtain $s.\dot{s} < 0$.

Proof:

In Table 6.1, the ZO-line corresponds to $s = 0$, $s > 0$ in the upper area and $s < 0$ in the lower, so Table 6.1 can be converted as

s	**NB**	**NS**	**ZO**	**PS**	**PB**
	PB	PS	ZO	NS	NB

Table 6.4: Typical Sliding-Mode Fuzzy Rules

and by Theorem 6.1, the consequent is determined on the basis of $s.\dot{s} < 0$.

Q.E.D.

Corollary 6.2: Sliding-Mode Fuzzy Control

On the basis of Corollary 6.1 and Proposition 6.2, a sliding-mode fuzzy control can be found from

$$u = \frac{\sum_{i=1}^{q} \mu_{S_i}(s).\hat{u}(\hat{s}_i)}{\sum_{i=1}^{q} \mu_{S_i}(s)}, \quad q \text{ rules} \tag{6.13}$$

We have the following theorem to design an integral robust sliding-mode control

Theorem 6.2: Integral SMC

If $\tilde{u}$ is a robust sliding-mode control, then an integral robust SMC can be determined by

$$u = \tilde{u} - \delta_i \int_0^t s(\tau).d\tau \tag{6.14}$$

where δ_i is a constant as an integral sliding margin .

Proof

Consider the following generic uncertain dynamical nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}).u \tag{6.15}$$

then

$$\dot{s}(t) = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).u$$

by Eq.(6.14), we have

$$\dot{s}(t) = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).\tilde{u} - \delta_i \mathbf{H}.\mathbf{g}(\mathbf{x})\int_0^t s(\tau).d\tau$$

or

$$s(t).\dot{s}(t) = s(t).\left[\mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).\tilde{u}\right] - \delta_i^+.s(t).\int_0^t s(\tau).d\tau \tag{6.16}$$

where

$$\delta_i^+ = \delta_i \mathbf{H}.\mathbf{g}(\mathbf{x}) \tag{6.16.a}$$

and choose

$$\delta_i = |\delta_i|.\mathrm{sgn}[\mathbf{H}.\mathbf{g}(\mathbf{x})] \;\Rightarrow\; \delta_i^+ = \delta_i \mathbf{H}.\mathbf{g}(\mathbf{x}) > 0 \tag{6.16.b}$$

Since $\tilde{u}$ is a robust SMC of Eq.(6.15), we have

$$s(t).[\mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).\tilde{u}] < 0$$

Consider the following 2 cases

- If $s(0) > 0$, then

$$\delta_i^+.s(\Delta t)\int_0^{\Delta t} s(\tau).d\tau > 0$$

where the sampling time Δt is chosen fast enough, so

$$s\dot{s}|_{\Delta t} = s(\Delta t).[\mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).\tilde{u}] - \delta_i^+.s(\Delta t).\int_0^{\Delta t} s(\tau).d\tau < 0$$

- If $s(0) < 0$, then

$$\delta_i^+.s(\Delta t)\int_0^{\Delta t} s(\tau).d\tau > 0$$

where the sampling time Δt is chosen fast enough, so

$$s\dot{s}|_{\Delta t} = s(\Delta t).[\mathbf{H}.\mathbf{f}(\mathbf{x}) + \mathbf{H}.\mathbf{g}(\mathbf{x}).\tilde{u}] - \delta_i^+.s(\Delta t).\int_0^{\Delta t} s(\tau).d\tau < 0$$

Q.E.D.

Based on Proposition 6.2 and Theorem 6.2, we have the following corollary to design a sliding-mode fuzzy control

Corollary 6.3: Sliding-Mode Fuzzy Control Design Rule

Table 6.5 can be used to design a stable sliding-mode fuzzy control. The δ is used for the reaching mode on the basis of the sliding-mode design rule, the larger δ the faster to reach the hyperplane. The gain δ_i is chosen large enough to eliminate of a possible steady-state error.

Proof:

In Table 6.5, the partial consequent of s satisfies Table 6.1, so the condition $s.\dot{s} < 0$ is obtained. In addition, the gain δ can make the antecedent $\delta\, s$ larger, so the larger consequent make $s \to 0$ faster to reach the hyperplane. The partial consequent of $\int s.dt$ contributes some negative to the consequent, by Theorem 6.2 the sliding mode is achieved and hence the system is stable.

$\frac{\delta_i \int s(t).dt}{\delta\, s(t)}$	$\mathbf{NB_i}$	$\mathbf{NS_i}$	$\mathbf{ZO_i}$	$\mathbf{PS_i}$	$\mathbf{PB_i}$
$\mathbf{NB_s}$	PB_u	PM_u	PS_u	PZ_u	ZO_u
$\mathbf{NS_s}$	PM_u	PS_u	PZ_u	ZO_u	NZ_u
$\mathbf{ZO_s}$	PS_u	PZ_u	ZO_u	NZ_u	NS_u
$\mathbf{PS_s}$	PZ_u	ZO_u	NZ_u	NS_u	NM_u
$\mathbf{PB_s}$	ZO_u	NZ_u	NS_u	NM_u	NB_u

Table 6.5: Typical Integral Sliding-Mode Fuzzy Rules

Q.E.D.

6.6. A NEW FUZZY IDENTIFICATION SCHEME

Consider the following *i*-th process rule of a step response

$$R^i: \textit{if response is } M^i, \textit{then } \dot{\mathbf{x}}^i = \mathbf{A}^i\mathbf{x} + \mathbf{B}^i u, \quad i = 1, \cdots, q; \quad q \text{ rules} \tag{6.17}$$

then a fuzzy model can be inferred from

$$\dot{\mathbf{x}} = \frac{\sum_{i=1}^{q} \mu_i\, \dot{\mathbf{x}}^i}{\sum_{i=1}^{q} \mu_i} \tag{6.18}$$

where

M^i is a fuzzy set of *response* in magnitude

μ_i is a membership function of magnitude M^i

Example:

$$R^1: \textit{if response } \text{is } \textit{small}, \textit{then } \dot{\mathbf{x}}^1 = \begin{bmatrix} 0 & 1 \\ a^1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ b^1 \end{bmatrix} u$$

$$R^2: \textit{if response } \text{is } \textit{medium}, \textit{then } \dot{\mathbf{x}}^2 = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ b^2 \end{bmatrix} u$$

$$R^3: \textit{if response } \text{is } \textit{big}, \textit{then } \dot{\mathbf{x}}^3 = \begin{bmatrix} 0 & 1 \\ a^3 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ b^3 \end{bmatrix} u$$

In a fuzzy control, a fuzzy inference is used to minimize an error between an actual system output and a desired system output. Differently, in a fuzzy identification, a fuzzy reasoning is used to infer a fuzzy model which includes a maximization of interactions between all the models. Thus we propose the following fuzzy identification

Proposition 6.3: Fuzzy Modelling

We get several step responses, both positive and negative if possible, and at different load for each step input if there is a variable load. On the basis of *magnitude* of responses, these mathematical models are fuzzified, inferred and defuzzified to obtain a fuzzy model where the uncertainty is taken into account

$$\dot{\mathbf{x}} = \frac{\sum_{i=1}^{q}\left[\prod_{j=1}^{q}\mu_j(\dot{\mathbf{x}}_i)\right].\dot{\mathbf{x}}_i}{\sum_{i=1}^{q}\left[\prod_{j=1}^{q}\mu_j(\dot{\mathbf{x}}_i)\right]} \quad \textbf{or} \quad \xi = \frac{\sum_{i=1}^{q}\left[\prod_{j=1}^{q}\mu_j(\xi_i)\right].\xi_i}{\sum_{i=1}^{q}\left[\prod_{j=1}^{q}\mu_j(\xi_i)\right]} \tag{6.19}$$

where ξ is any element of **A** or **B** in Eq.(6.18).

6.7. Practical System Identification

Most of physical systems can be practically decomposed into some subsystems of first order or second order. In this section, we will review some relevant step responses of first and second order systems for convenience to identify a rough mathematical model. First we revise some results in Laplace transform which will be used in this section

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\left[s\,F(s)\right]$$

Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[s\,F(s)\right]$$

A step response of amplitude a and its derivative will be used to identify the dynamics. The derivative helps to identify 2 similar step responses and to recognize a possible time-delay in a step response.

6.7.1. Identification of Systems with Negative Real Poles

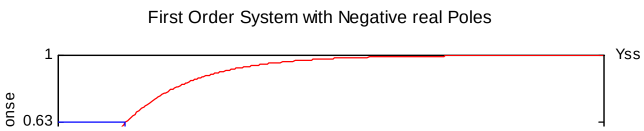


Fig. 6.20: $G(s) = \dfrac{K}{s+a}, \quad K = \dfrac{a\,Y_{ss}}{A}$

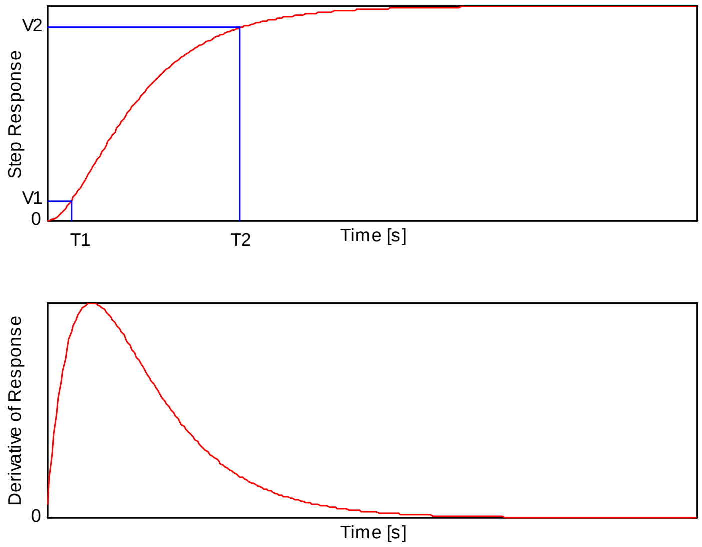


Fig. 6.21: $G(s)=\dfrac{K}{(s+a)(s+b)}$, $\quad K=\dfrac{a\,b\,Y_{ss}}{A}$

where a, b are solutions of the following equation using numerical method with 2 points (T_1,V_1) and (T_2,V_2) on the derivative of step response

$$\begin{cases} e^{-a\,T_1}-e^{-b\,T_1}=\dfrac{b-a}{a\,b\,Y_{ss}}V_1 \\ e^{-a\,T_2}-e^{-b\,T_2}=\dfrac{b-a}{a\,b\,Y_{ss}}V_2 \end{cases} \Rightarrow \begin{cases} e^{-a\,T_1}-e^{-b\,T_1}=\left(\dfrac{1}{a}-\dfrac{1}{b}\right)\dfrac{V_1}{Y_{ss}} \\ e^{-aT_2}-e^{-b\,T_2}=\left(\dfrac{1}{a}-\dfrac{1}{b}\right)\dfrac{V_2}{Y_{ss}} \end{cases}$$

Since the equation are transcendental, we may need the Taylor series to obtain an approximate algebraic equations to ease the problem

$$\begin{cases} T_1-\dfrac{1}{2}(a+b)T_1^2+\dfrac{1}{6}\left(a^2+ab+b^2\right)T_1^3-\dfrac{1}{24}\left(a^3+a^2b+ab^2+b^3\right)T_1^4=\dfrac{V_1}{a\,b\,Y_{ss}} \\ T_2-\dfrac{1}{2}(a+b)T_2^2+\dfrac{1}{6}\left(a^2+ab+b^2\right)T_2^3-\dfrac{1}{24}\left(a^3+a^2b+ab^2+b^3\right)T_2^4=\dfrac{V_2}{a\,b\,Y_{ss}} \end{cases}$$

Proposition 6.4: Identification of Over-Damped Second Order System

Instead of solving the above equations, since a/b is not larger than 3, otherwise the system can be considered as a first order system, we can obtain a rough model using a graphical method by starting with α based on the time-constant of an approximate first order system.

6.7.2. Identification of System with Integral Element

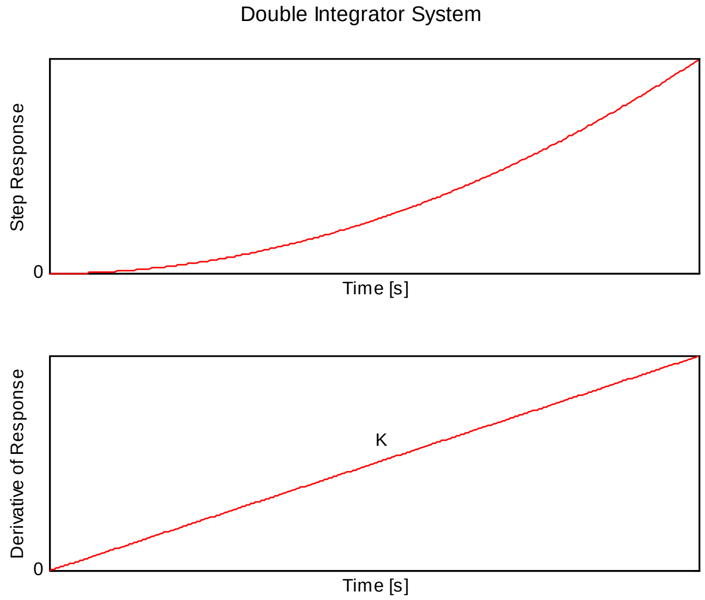


Fig. 6.22: $G(s) = \dfrac{K}{s^2}$

where K is the slop of the line in the derivative of the step response.

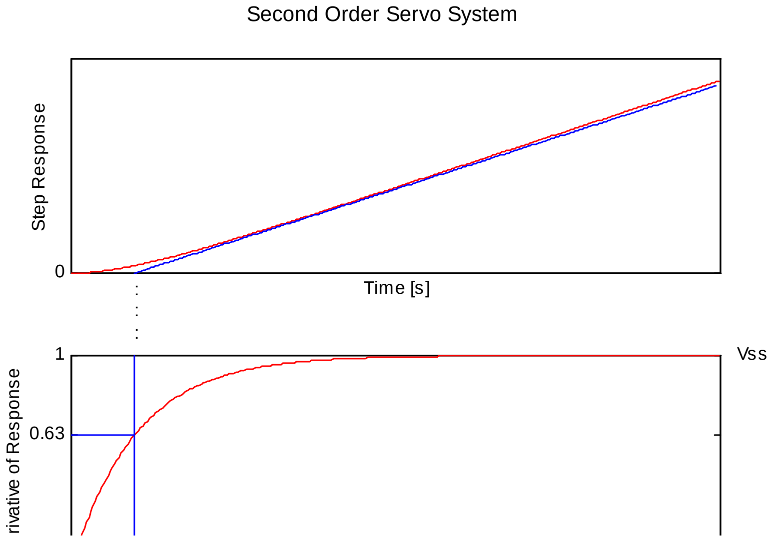


Fig. 6.23: $G(s) = \dfrac{K}{s(s+a)}, \qquad K = \dfrac{a\,V_{ss}}{A}$

Remark 6.10: Necessity of Derivative Response

- The step responses in Figs. 6.20 & 6.21 look similar, but the difference is in their derivatives. Similarly for Figs. 6.22 & 6.23. This justifies the necessity of a derivative of response.
- If the steady-state value of derivative is not available, the time-constant can be estimated using the asymptote in the step response as in Fig. 6.23.

6.7.3. Identification of Second Order System under Sustaining Oscillation

In this sub-section, y_c at the center of oscillation instead of y_{ss}, we will use the period T determined on the step response graph, y_a is an amplitude of a sinusoid and

$$\omega = \frac{2\pi}{T}$$

6.7.3.1. CASE OF $y_0 = 0,\ y_c \neq 0$

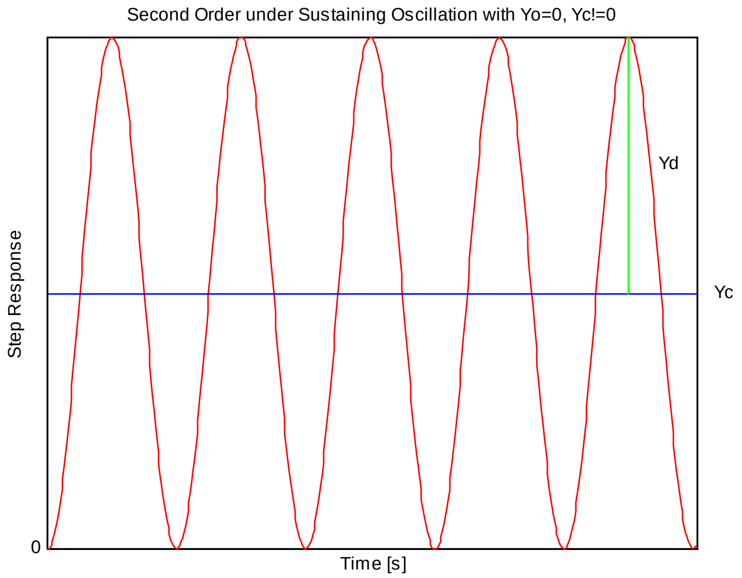


Fig. 6.24: $G(s) = \dfrac{K}{s^2 + \omega^2}, \qquad K = \dfrac{\omega^2\, y_a}{a}$

6.7.3.2. CASE OF $y_0 = 0,\ y_c = 0$

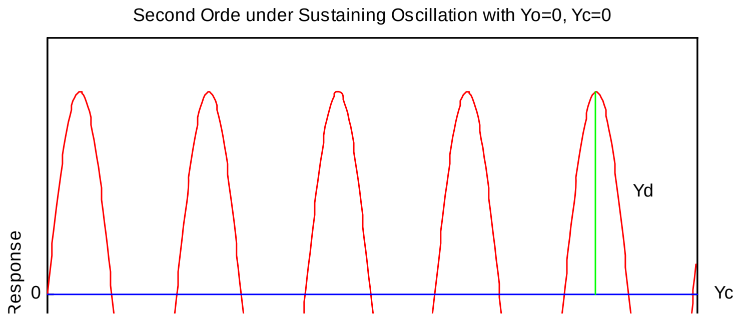


Fig. 6.25: $G(s) = \dfrac{K\, s}{s^2 + \omega^2}, \qquad K = \dfrac{\omega\, y_a}{a}$

6.7.3.3. CASE OF $y_0 \neq 0,\ \ y_c = 0$

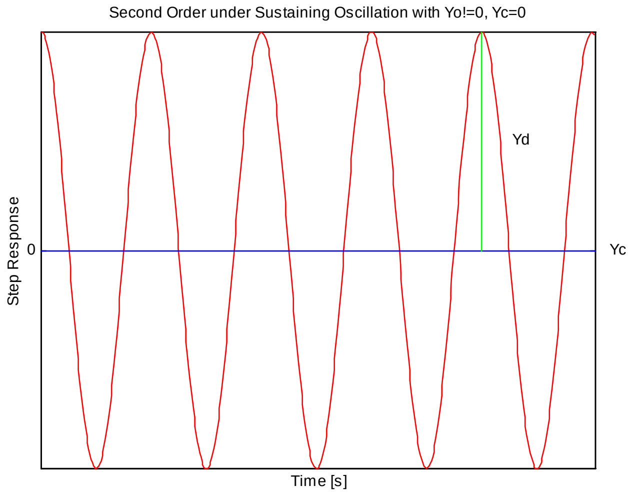


Fig. 6.26: $G(s) = \dfrac{K\,s^2}{s^2 + \omega^2}, \qquad K = \dfrac{y_a}{a}$

6.7.4. Identification of Second Order under Damped Oscillation

In this sub-section, we will use

$$\beta = \sqrt{1 - \zeta^2}$$

and

$$\omega = \frac{2\pi}{T} \ \Rightarrow\ \omega_n = \frac{\omega}{\beta}$$

and

$$\delta_* = \frac{\delta_1}{\delta_2} \ \Rightarrow\ \zeta = \frac{1}{\sqrt{1 + (2\pi/\ln\delta_*)^2}}$$

where T, δ_1, δ_2 will be determined on the step response graph.

6.7.4.1. CASE OF $y_0 = 0,\ \ y_{ss} \neq 0$

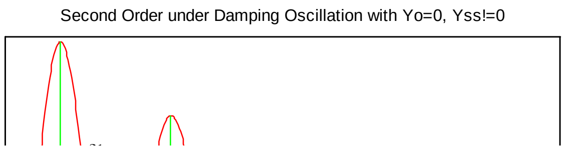


Fig. 6.27: $G(s) = \dfrac{K}{s^2 + 2\zeta\omega_n + \omega_n^2}, \quad K = \dfrac{Y_{ss}\,\omega_n^2}{a}$

6.7.4.2. CASE OF $y_0 = 0,\ \ y_{ss} = 0$

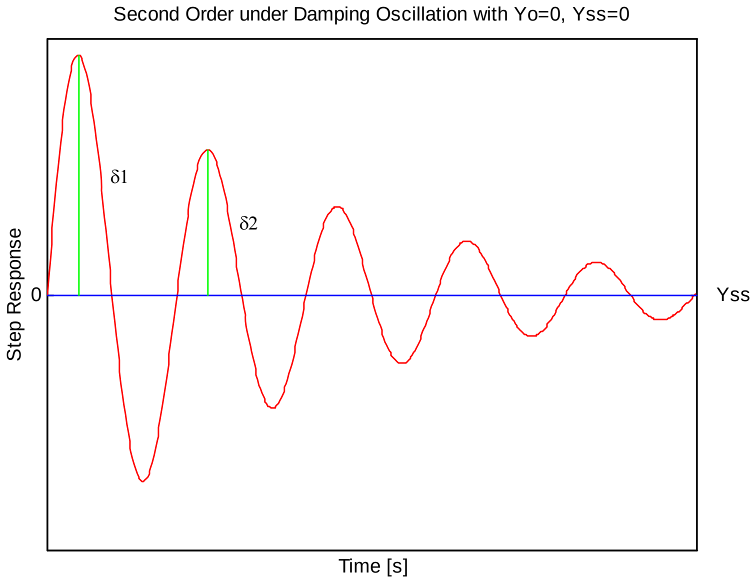


Fig. 6.28: $G(s) = \dfrac{K\,s}{s^2 + 2\zeta\omega_n + \omega_n^2},\qquad K = \dfrac{\omega\delta_1}{a}\exp\left(\dfrac{\pi\,\zeta}{2\beta}\right)$

6.7.4.3. CASE OF $y_0 \neq 0,\ \ y_{ss} = 0$

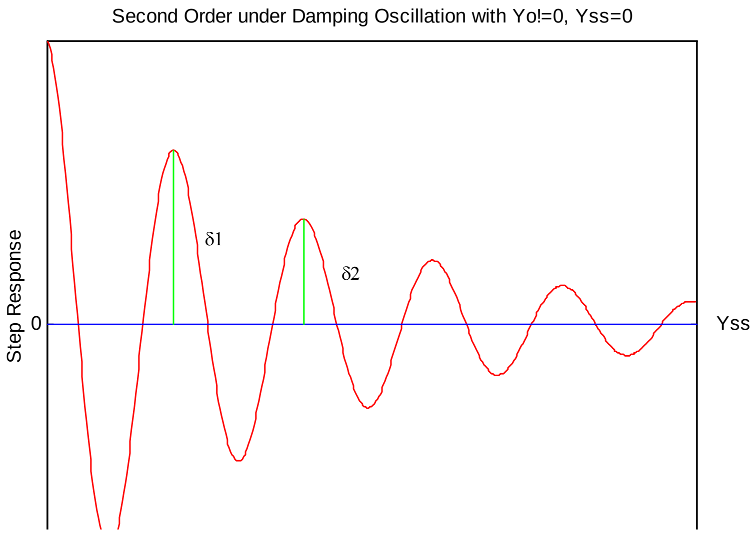


Fig. 6.29: $G(s) = \dfrac{K\,s^2}{s^2 + 2\zeta\omega_n + \omega_n^2},\qquad K = \dfrac{y_0}{a}$

6.8. Case Studies of Fuzzy Identifications

An 2-nd order Static-VAR and a 3-rd order Ball-Hoop systems will be considered in this section.

6.8.1. Static-VAR System

We will obtain several step responses with different step inputs, since the system dynamics are highly uncertain due to the loading, different loads are set for each step input.

6.8.1.1. System Order Identification

First we identify the system order by considering one of step responses

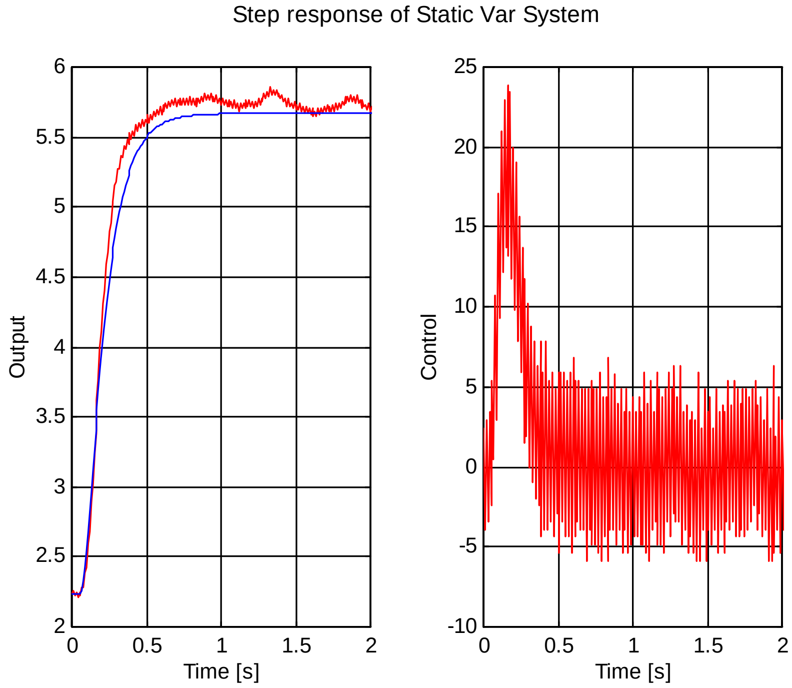


Fig. 6.30: Step Response for Static-VAR System

The derivative of step response shows that the system can be considered as a second order system with a time delay of 0.05 sec.

6.8.1.2. Fuzzy Identification

The followings are step responses of different step inputs at different loads, they are sorted in ascending order of magnitude.

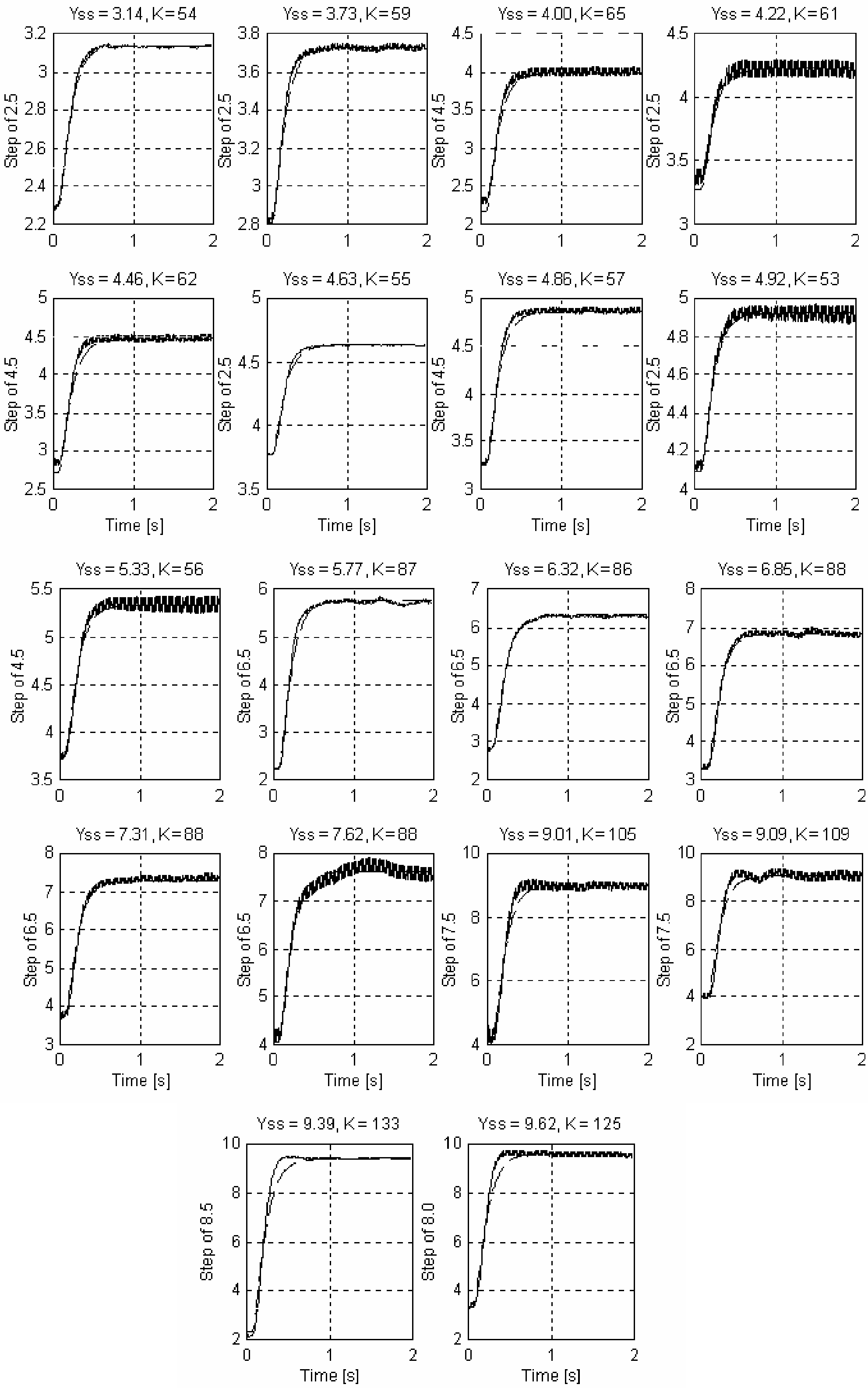


Fig. 6.31: Step Responses of Static-VAR System

By Proposition 6.3, we have 18 mathematical models above of the following forms

$$G(s)=\frac{K\,e^{-0.05s}}{(s+8)(s+20)}$$

where K specified at each plot in Fig. 6.31 above.

These models are fuzzified, inferred and defuzzified on the basis of steady-state of step response specified in each plot in Fig. 6.31 above to obtain the following fuzzy model

$$G(s)=\frac{K\,e^{-0.05s}}{(s+8)(s+20)}, \qquad K=87.3\in[53,133]$$

6.8.2. Ball-Hoop System

The Ball-Hoop system is composed of 2 subsystems: hoop and ball subsystems.

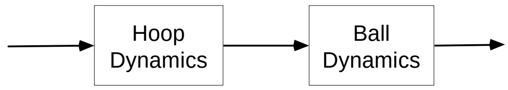


Fig. 6.32: Ball-Hoop System

6.8.2.1. System Order Identification

- Hoop Dynamics

Using a step of 2, we have the following step response for hoop dynamics

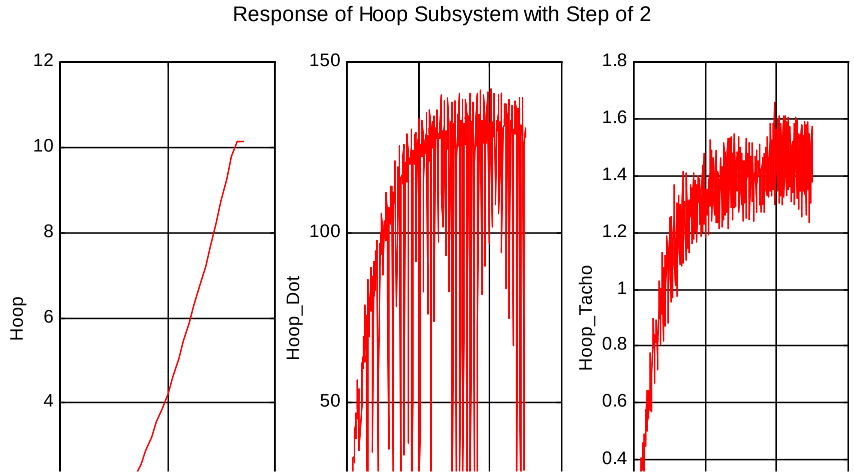


Fig. 6.33: Response of Hoop Subsystem with step of 2

where Hoop_Dot is the difference of consecutive elements of Hoop, *ie.*

$$Hoop\_Dot[i]=\frac{Hoop[i+1]-Hoop[i]}{T_s}$$

We need to filter out all noise before modelling. In doing so, we use a low-pass digital filter FIR (Finite-Duration Impulse Response) with cut-off frequency at 3 Hz, Kaiser window with β=3, filter length of 21.

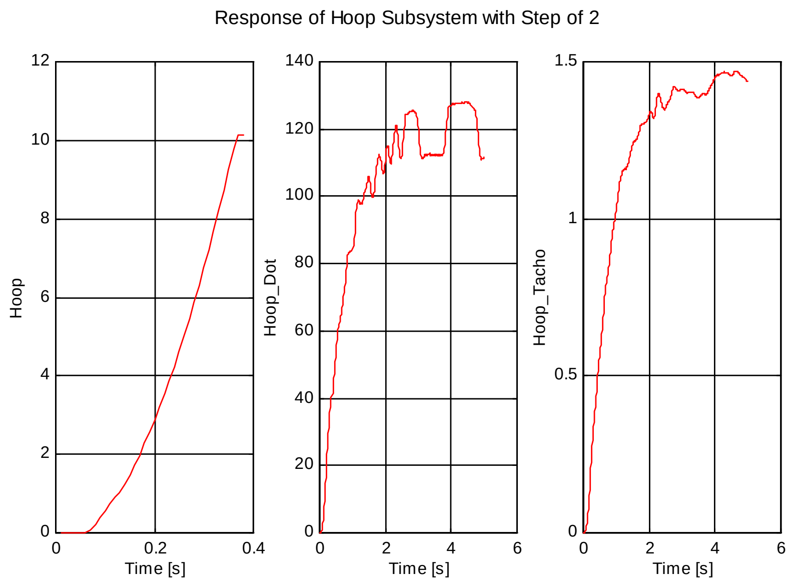


Fig. 6.34: Response of Hoop Subsystem with step of 2 after Low-Pass Digital Filter FIR using Kaiser window with $f_c = 3Hz,\ \beta = 3,\ L = N$.

From the graph of Hoop_Dot, we have the following second order model for the hoop dynamics

$$G_h(s) = \frac{K}{s(s+\alpha)}$$

- Ball Dynamics

We have the following step response of the ball dynamics

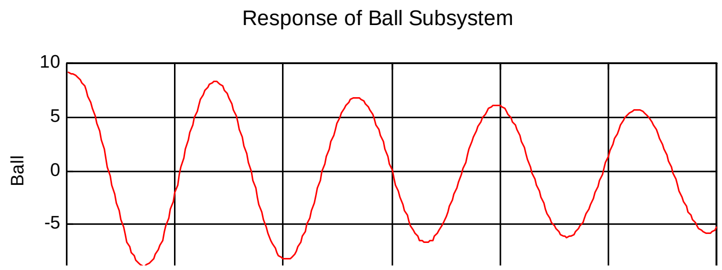


Fig. 6.35: Step Responses of Ball Subsystem with Negative Step Input

so we have the following second order model for the ball dynamics

$$G_b(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

thus we have the following model of Ball-Hoop system

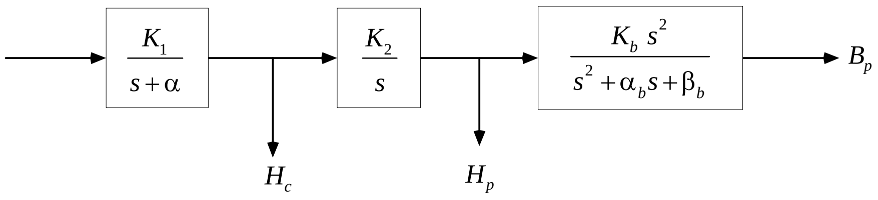


Fig. 6.36: Identification of Ball-Hoop System Model

6.8.2.2. Fuzzy Identification

9 models have been achieved from experiments. Since the hoop position will be under control, the 9 models will be in ascending order of the magnitudes of the final values from the hoop position where all responses are normalized using unit step input.

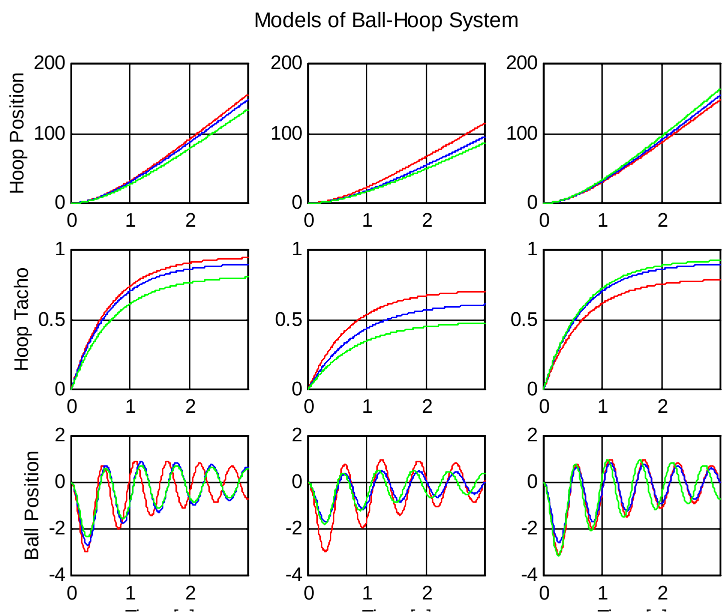


Fig. 6.37: Step Responses for 9 Models of Ball-Hoop System

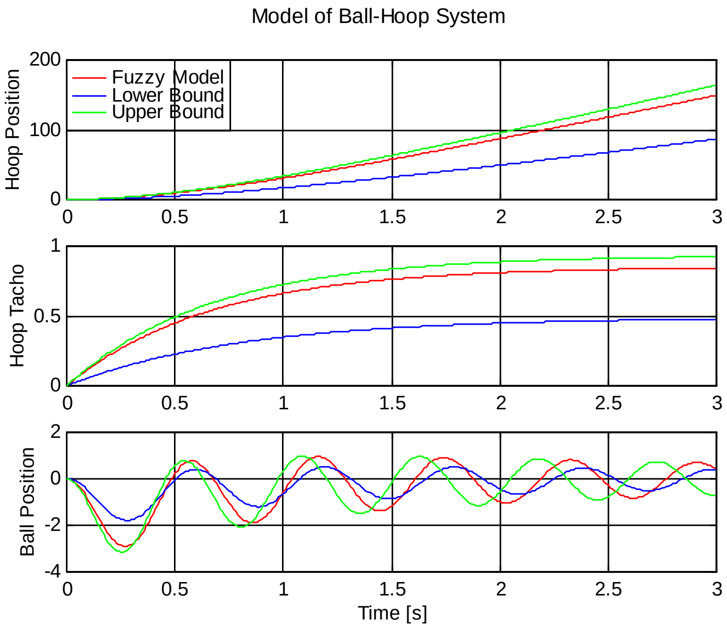


Fig.6.38: Fuzzy Model of Ball-Hoop System

By Proposition 6.3, we have the following fuzzy model of Ball-Hoop system

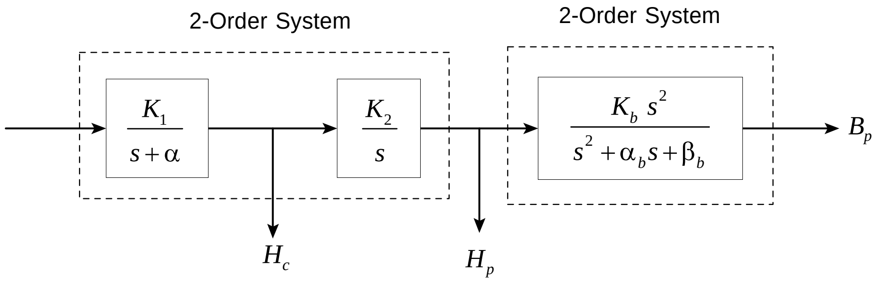


Fig. 6.39: Fuzzy Model of Ball-Hoop System

where

$$K_1 = 1.28 \in [0.61, 1.43], \quad K_2 = 74.92 \in [70, 80], \quad \alpha = 1.5 \in [1.2, 1.5]$$

$$K_b = -2.25 \in [-2.0, -2.7], \quad \alpha_b = 0.65 \in [0.6, 0.69], \quad \beta_b = 115.74 \in [100, 132]$$

and

H_p, B_p are hoop and ball position

H_c is hoop tacho

Remark 6.11: Higher-Order System decomposed into 2-Order Subsystems

4-order Ball-Hoop system can be decomposed into 2 second-order subsystems as in Fig. 6.38.

6.9. NUMERICAL EXAMPLES

Fuzzy structure is based on Proposition 6.2. For sliding-mode fuzzy controls, sliding and integral sliding margin are determined on the basis of the sliding-mode fuzzy design rule proposed in Corollary 6.3. Gains of $e, \dot{e}$ are arbitrarily chosen unity for conventional fuzzy controls.

6.9.1. Example 6.1: Fast Servo Motor System

Consider the fast servo motor system

$$\mathbf{A} = \begin{bmatrix} 0, & 1 \\ 0, & -20 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -600 \end{bmatrix}$$

6.9.1.1. Fuzzy Control with Unity Gains

From the section above, the best result for this system has used the following fuzzy parameters

$$\mathbf{e} = [-2.0000, \ -0.6667, \ 0, \ 0.6667, \ 2.0000]$$
$$\mathbf{d} = [-5.0000, \ -1.6667, \ 0, \ 1.6667, \ 5.0000]$$
$$\mathbf{u} = [-10, \ -5, \ 0, \ 5, \ 10]$$

then

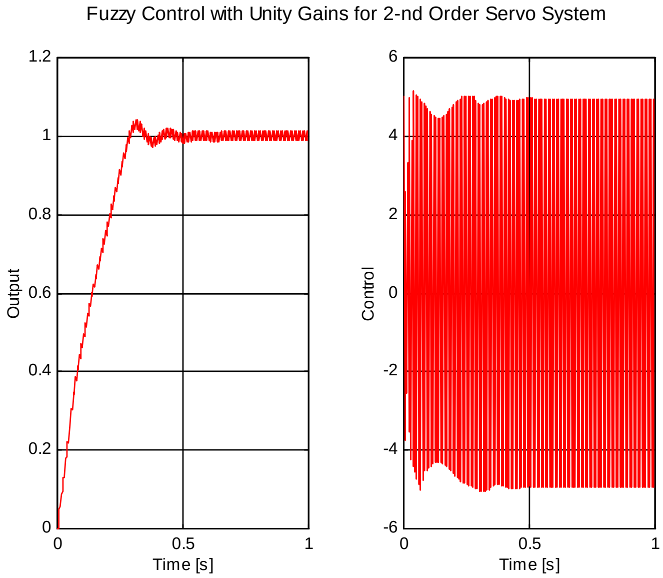


Fig. 6.40: Conventional Fuzzy Control using Soft Rulebase of 5 Rules with Weighted Input and Normal Output for Example 6.1 with unity gain for $e, \dot{e}$: **chattering**

6.9.1.2. Sliding-Mode Fuzzy Control

By the design rule, choose

$$\lambda_H = -12, \ \delta = 10, \ \delta_i = 1$$

and

$$\mathbf{s} = [-2.0000, \ -0.6667, \ 0, \ 0.6667, \ 2.0000]$$
$$\mathbf{s}_i = [-1.0000, \ -0.3333, \ 0, \ 0.3333, \ 1.0000]$$
$$\mathbf{u} = [-10, \ -5, \ 0, \ 5, \ 10]$$

then we have

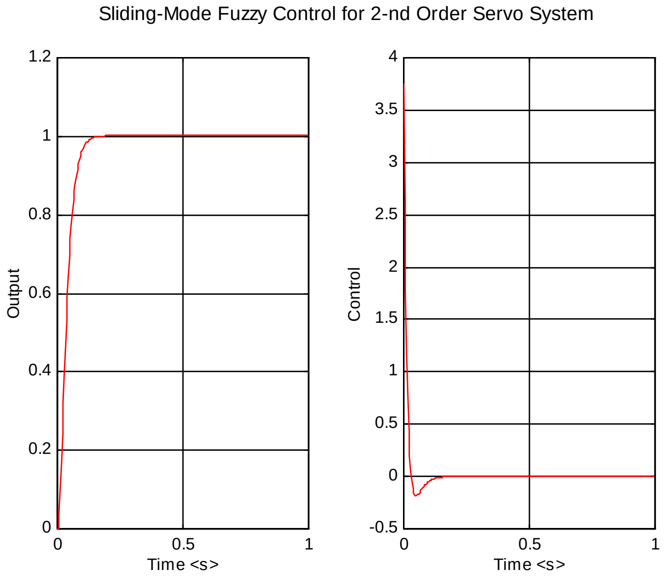


Fig. 6.41: Sliding-Mode Fuzzy Control for Example 6.1: **No Chattering**

6.9.2. Example 6.2: Static VAR System

Consider the static VAR system above

$$\dot{\mathbf{x}}(t)=\mathbf{A}.\mathbf{x}(t)+\left(\mathbf{B}+\Delta\tilde{\mathbf{B}}\right).u(t-T_d),\quad \mathbf{A}=\begin{bmatrix}0, & 1\\ -160, & -28\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\ 87\end{bmatrix},\quad \Delta\mathbf{B}=\begin{bmatrix}0\\ 46\end{bmatrix},\quad T_d=0.05$$

so with the sampling time of $T_s=0.01\,s$, we have the following discrete model

$$\mathbf{x}(k+1)=\mathbf{A}_d\mathbf{x}(k)+\left(\mathbf{B}_d+\Delta\tilde{\mathbf{B}}_d\right).u(k-d),\quad \mathbf{A}_d=\begin{bmatrix}0.9927, & 0.0087\\ -1.3918, & 0.7491\end{bmatrix},\quad \mathbf{B}_d=\begin{bmatrix}0.0040\\ 0.7568\end{bmatrix},\quad \Delta\mathbf{B}_d=\begin{bmatrix}0.0021\\ 0.4001\end{bmatrix},\quad d=\frac{T_d}{T_s}=5$$

or

$$\mathbf{x}(k+d+1)=\mathbf{A}_d\mathbf{x}(k+d)+\mathbf{B}_d u(k),\quad \mathbf{A}_d=\begin{bmatrix}0.9927, & 0.0087\\ -1.3918, & 0.7491\end{bmatrix},\quad \mathbf{B}_d=\begin{bmatrix}0.0040\\ 0.7568\end{bmatrix},\quad \Delta\mathbf{B}_d=\begin{bmatrix}0.0021\\ 0.4001\end{bmatrix},\quad d=\frac{T_d}{T_s}=5$$

thus

$$u(k)=\phi\left[\mathbf{x}(k+d)\right]$$

and

$$s(k)=\psi\left[\mathbf{x}(k+d)\right]$$

6.9.2.1. Fuzzy Control with Unity Gains

Choose fuzzy parameters as

$$\mathbf{e}=\left[-4,\ -2,\ 0,\ 2,\ 4\right]$$
$$\mathbf{d}=\left[-10,\ -5,\ 0,\ 5,\ 10\right]$$
$$\mathbf{u}=\left[-10,\ -5,\ 0,\ 5,\ 10\right]$$

then

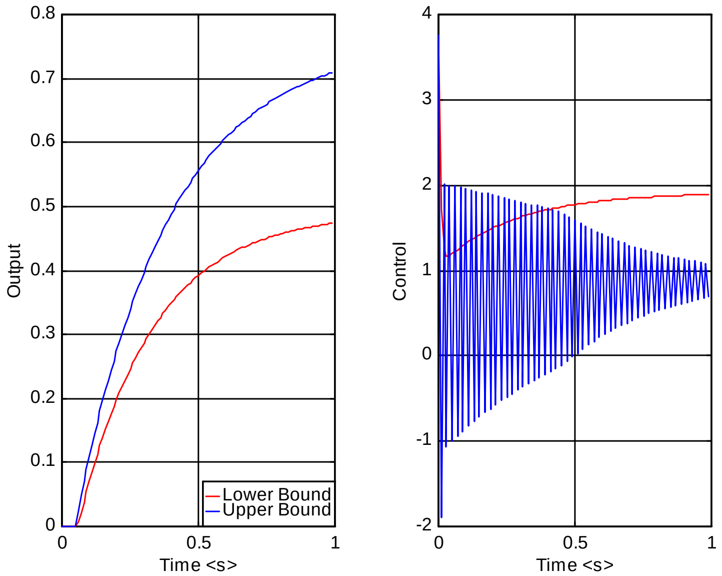


Fig. 6.42: Conventional Fuzzy Control for Example 6.3: Significant Steady-State Error

6.9.2.2. Robust Sliding-Mode Fuzzy Control

Choose the same fuzzy parameters above and by the design rule, choose

$$\lambda_H = -20, \quad \delta = 12, \quad \delta_i = 50$$

then

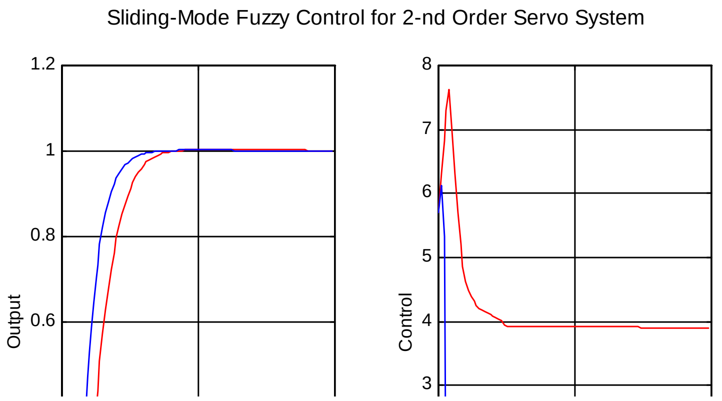


Fig. 6.43: Robust Sliding-Mode Fuzzy Control for Example 6.3.

Remark 6.12: Sliding-Mode Fuzzy Control Invariant to Fuzzy Structure

In Example 6.1 and 6.2, varying the rule number and fuzzification weighted yields the same response.

6.9.3. Example 6.3: Ball-Hoop System

Since it is not convenient to use more than 2 entries in a typical fuzzy rulebase, a convention fuzzy control may be hardly applied for systems higher than third order unless the system can be decomposed into some 2-nd order sub-systems. We consider a robust sliding-mode fuzzy control in this example of 4-th order system.

The fuzzy model of ball-hoop is used as the nominal model since it is the most potential model:

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 75 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1.5 & 0 \\ 0 & -116 & 252 & -0.65 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.67 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 16.3 & 84.4 & 0.046 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.28 \\ -215 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.67 \\ 130 \end{bmatrix}$$

by the design rule, choose

$$\lambda_H = [-8, \quad -8, \quad -8], \quad \delta = 8, \quad \delta_i = 0.1$$

thus

$$\mathbf{H} = [0.0460 \quad -0.0862 \quad 1.2738 \quad 0.0029]$$

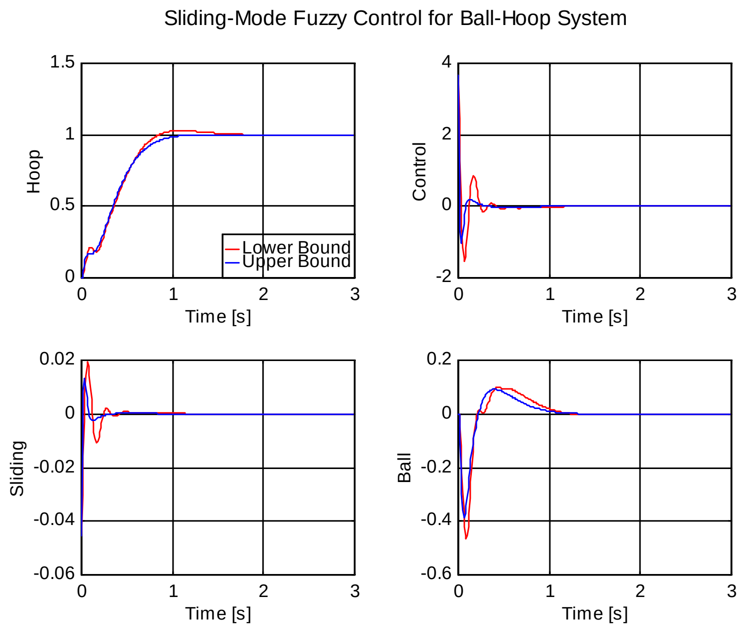


Fig. 6.44: Sliding-Mode Fuzzy Control with New Fuzzy Identification for Example 6.4.

6.10. CONCLUSION

The stability of a fuzzy control is not based on the fuzzy model so it is applicable to a conventional fuzzy control where a fuzzy model is not available. The proposed fuzzy identification is used to obtain a fuzzy model for the purpose of the hyperplane design.

A conventional fuzzy control has no problem with slow systems; however, for a fast system there may be the chattering problem since the system dynamics are not included in the design, that is the gains for error and its change are unity regardless the system dynamics. A fuzzy control using the soft rulebase with 5 rules of normal or weighted input and normal output triangular defuzzifications can be best applied for both slow and fast systems.

The performance of a sliding-mode fuzzy control is better than that of a conventional one since the system dynamics are included in the design of a sliding-mode fuzzy control, and varying the rule number and fuzzification weighting yields no remarkable improvement.

It is not convenient to introduce more than 2 input variables in a fuzzy table, normally the first entry is for error and the second is for change of error or sum of error. It may be hard to use a conventional fuzzy control for systems higher than second order unless the system can be decomposed into some 2-nd order sub-systems. In addition, for a 2-nd order system, the error must be used as the first entry in a fuzzy table; however, only change of error or sum of error can be used in the second entry, but not both. By using a sliding mode fuzzy control, this limitation is removed because all system states can be included in a sliding variable via a hyperplane equation (derivative action) as the first entry of a fuzzy table, and a sum of error can be used as the second entry. The scheme using sum of s without integration of error may be advantage over change of s with integration of error.

Chapter 6

Robust Sliding-Mode Fuzzy Controller Design

General Sliding-Mode Controller Design for MIMO Uncertain Nonlinear Systems

7.1. INTRODUCTION

In this chapter, we present a sliding-mode control (SMC) design for SISO and MIMO nonlinear systems. So far we have only considered the case where the output is the first system state. In this chapter we will consider the *general* case where the output is a nonlinear function of all system states.

For a MIMO SMC, the hierarchical control technique has been used in Utkin 1977 for linear systems. Alternatively, we will use a decoupling technique which is applicable for MIMO nonlinear systems. This technique allows a MIMO to be considered as a set of SISO subsystem. As consequence, all SISO results developed so far can be applied, including this chapter.

For the general SMC design for MIMO nonlinear systems, in the SMC literature (Fernandez *et al.* 1987; Chen *et al.* 1992), the hyperplane design has been based on the *Input-Output Linearization* technique (Hunt *et al.* 1983, Isidori 1985, Kravaris *et al.* 1986) to transform a nonlinear system into a canonical nonlinear system. By the nature of a hyperplane that it is of reduced-order, we will design the hyperplane via the direct allocation approach in Chapter 2. The controller will be designed in a unified manner as in other cases. The proposed robust design may be the simplest approach in the literature (Fu 1992, Sira-Ramirez 1996). In addition, the design scheme in Proposition 3.3 is proved to be efficient in solving the chattering and steady-state error (Example 7.6).

Part A : SMC Design for SISO Uncertain Nonlinear Systems

We start with the hyperplane design and a *nonlinear* system stability test. Next we present a *discontinuous* SISO nonlinear SMC and a *continuous* SISO nonlinear SMC. We then present a general SMC design for nonlinear systems where the output does not have to be the first state, $y \neq x_1$, it can be a nonlinear function of system states, $y = \eta(\mathbf{x})$. We then present a robust discontinuous SMC design and robust continuous Pseudo-SMC design for SISO uncertain nonlinear systems.

7.2. HYPERPLANE AND STABILITY

In Chapter 2 on the hyperplane design, the proof of the direct allocation method is based on linear systems. In Chapter 3, the proof of the stability test is based on linear systems. Now both proofs will be modified so that they are also applicable to nonlinear systems. As consequence, the I/O-state technique can be applicable for both linear and nonlinear systems.

7.2.1. Hyperplane Design

Theorem 7.1: Hyperplane Design for Canonical Nonlinear Systems

For a n-ordered canonical *linear or nonlinear* system, if the hyperplane-eigenvalues are

$$\boldsymbol{\lambda}_H = \left[\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_{n-1}\right] \tag{7.1}$$

then a hyperplane can be found by

$$s = \mathbf{H}.\mathbf{x}, \quad \mathbf{H} = \left[h_1 \quad h_2 \quad \cdots \quad h_{n-1} \quad 1\right] \tag{7.2}$$

where h_i is the coefficients of the following polynomial

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{n-1}) = \lambda^{n-1} + h_{n-1}\lambda^{n-2} + \cdots + \lambda_2 p + \lambda_1 \tag{7.3}$$

Proof

Consider the following n-ordered canonical linear or nonlinear system of the output $y = x_1$

$$\begin{cases} \dot{y} = \dot{x}_1 = x_2 \\ \ddot{y} = \dot{x}_2 = x_3 \\ \cdots \\ y^{(n-1)} = \dot{x}_{n-1} = x_n \\ y^{(n)} = \dot{x}_n = \phi(\mathbf{x}), \quad \phi(\mathbf{x}) : \text{linear or nonlinear function of the system state variable } \mathbf{x} \end{cases}$$

and a hyperplane

$$s = \mathbf{H}\mathbf{x} = \left[h_1, \quad \cdots, \quad h_{n-1}, \quad 1\right].\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = h_1 x_1 + \cdots + h_{n-1}x_{n-1} + x_n$$

In the sliding mode, we have the following linear differential equation

$$s = 0 \;\Rightarrow\; h_1 x_1 + \cdots + h_{n-1}x_{n-1} + x_n = 0 \;\Rightarrow\; h_1 y + \cdots + h_{n-1}y^{(n-2)} + y^{(n-1)} = 0 \tag{7.4}$$

then the corresponding characteristic equation is

$$\lambda^{n-1} + h_{n-1}\lambda^{n-2} + \cdots + h_1 = 0 \tag{7.5}$$

thus if the roots of this equation Eq.(7.5) are Hurwitz, then the output is settled down via the above linear differential equation Eq.(7.4).

Therefore, to design a hyperplane for a canonical linear or nonlinear system, first choose the $(n-1)$ desired hyperplane-eigenvalues to be Hurwitz

$$\boldsymbol{\lambda}_H = \left[\lambda_1, \quad \cdots, \quad \lambda_{n-1}\right] \subset \Re_{(-)}$$

then the hyperplane is determined by the coefficients of the characteristic equation as follows

$$(\lambda - \lambda_1)\cdots(\lambda - \lambda_{n-1}) = \lambda^{n-1} + h_{n-1}\lambda^{n-2} + \cdots + h_2\lambda + h_1$$

Q.E.D.

Remark 7.1: Hyperplane Design for Canonical Linear and Nonlinear Systems

- Note that this proof is more general than that in Section 2.3.1 because it is for both linear and nonlinear systems.
- Note that $\phi(\mathbf{x})$ is not taken into account in the hyperplane design, but it will be in the controller design. This is consistent with the invariance condition because a canonical form satisfies the matching condition.

7.2.2. SMC Stability Criterion for Nonlinear Systems

Based on Theorem 7.1 above, we have the following corollary on a stability test for both linear and nonlinear systems.

Corollary 7.1: Stability Criterion

For an n-ordered canonical *linear or nonlinear* system, if there exists a control function to satisfy the sliding condition $s.\dot{s} \le 0$ on the hyperplane determined by Theorem 7.1, then the system is stable.

Proof

The control function satisfying the sliding condition will drive the system into the sliding mode. Then, by Theorem 7.1, the system is stable.

Q.E.D.

7.3. Robust Discontinuous SMC Design for SISO Uncertain Nonlinear Systems

In this section, we will consider a robust SMC design for SISO uncertain nonlinear systems.

7.3.1. Discontinuous SMC Design for SISO Nonlinear Systems

We first define a *nonlinear system states vector* to be used in a SMC for a nonlinear system

Definition 7.1: Nonlinear System States Vector

Consider a SISO nonlinear system

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t).u \tag{7.6}$$

then nonlinear system states vector is composed of *every term* in $\mathbf{f}(\mathbf{x},t)$.

For example in Example 7.1, we have

$$\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}.u, \qquad \mathbf{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1) \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 1 + \sqrt{|x_1|} \end{bmatrix}$$

then

$$\mathbf{w} = \begin{bmatrix} x_2, & x_1x_2, & x_1^2, & \sin(tx_1) \end{bmatrix}$$

We then have the following theorem to design a discontinuous SMC for a nonlinear system

Theorem 7.2: Discontinuous SMC Design for Nonlinear Systems

Consider a SISO nonlinear system in Eq.(7.6), then a discontinuous SMC control function can be found by

$$u = u_e + u_r \tag{7.7}$$

where

- equivalent control

$$u_e = -(\mathbf{Hg})^{-1}\mathbf{Hf} = -\mathbf{K}_e.\mathbf{w}, \quad \mathbf{w} = \mathbf{w}(\mathbf{x}) = \left|w_i(\mathbf{x})\right\rangle \tag{7.7.a}$$

- reaching control

$$u_r = -(\mathbf{Hg})^{-1}\mathbf{K}_r.|\mathbf{w}|.\mathrm{sgn}(s), \quad \mathbf{K}_r = \left\langle\delta\right| \tag{7.7.b}$$

with

$$\mathbf{x},\mathbf{f},\mathbf{g} \in \Re^{n\times 1}; \quad \mathbf{H} \in \Re^{1\times \mathrm{n}}; \quad u,s \in \Re; \quad \delta \in \Re_{(+)} \text{ : sliding margin.}$$

Proof

Consider a hyperplane

$$s = \mathbf{Hx} \Rightarrow \dot{s} = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}(\mathbf{f}+\mathbf{g}.u) = \mathbf{Hg}.\left[u + (\mathbf{Hg})^{-1}\mathbf{Hf}\right] = \mathbf{Hg}.\left(u - u_{eq}\right)$$

then the definition of $\mathbf{w}$ above, we have

$$-u_{eq} = (\mathbf{Hg})^{-1}\mathbf{Hf} = \sum k_{ei} \times w_i(\mathbf{x})$$

As usual, let

$$u = \sum k_i \times w_i(\mathbf{x})$$

then, to satisfy the sliding condition, we obtain

$$s\dot{s} < 0 \Rightarrow k_i \begin{cases} < -k_{ei}, & \text{if} \quad s\mathbf{Hg}w_i > 0 \\ > -k_{ei}, & \text{if} \quad s\mathbf{Hg}w_i < 0 \end{cases} \Rightarrow k_i = \begin{cases} -k_{ei} - \delta^*, & \text{if} \quad s\mathbf{Hg}w_i > 0 \\ -k_{ei} + \delta^*, & \text{if} \quad s\mathbf{Hg}w_i < 0 \end{cases}$$

thus by Lemma 3.1, a discontinuous SMC control function is

$$u = -\sum k_{ei} \times w_i(\mathbf{x}) - \sum \delta^* \times \left|w_i(\mathbf{x})\right|.\mathrm{sgn}(s) = -\sum k_{ei} \times w_i(\mathbf{x}) - (\mathbf{Hg})^{-1}\sum \delta \times \left|w_i(\mathbf{x})\right|.\mathrm{sgn}(s)$$

where

$$\delta = \mathbf{Hg}.\delta^*$$

Q.E.D.

Remark 7.2: Reaching Control

We have chosen equal sliding margin for the reaching control since we have not known the dynamics of **w**. This remark will applies for all designs in the sequel.

7.3.2. Robust Discontinuous SMC Design for SISO Uncertain Nonlinear Systems

We have the following assumption

Assumption 7.1: System Constraint on Parametric Variation

A system matrix g takes any variation such that the polarity of $(\mathbf{H}\tilde{\mathbf{g}})$ is unchanged, i.e.

$$\mathrm{sgn}(\mathbf{H}\tilde{\mathbf{g}}) = \mathrm{sgn}(\mathbf{Hg}) \tag{7.8}$$

where $\mathbf{g}$ is the nominal value of $\tilde{\mathbf{g}}$ **.**

Under Assumption 7.1, we have the following theorem

Theorem 7.3: Robust Discontinuous SMC Design for Uncertain Nonlinear Systems

Consider a canonical uncertain nonlinear SISO system

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{\mathbf{g}}(\mathbf{x},t).u \tag{7.9}$$

then, under Assumption 7.1, a robust discontinuous SMC control function is determined by

$$\underline{\underline{u = u_e + u_r + u_p}} \tag{7.10}$$

where

- equivalent control

$$u_e = -\mathbf{K}_e.\mathbf{w}, \quad \mathbf{K}_e = \left\langle \frac{\sup_i + \inf_i}{2} \right| \tag{7.10.a}$$

- reaching control

$$u_r = -\mathbf{K}_r.|\mathbf{w}|.\operatorname{sgn}(s\mathbf{H}\overline{\mathbf{g}}), \quad \mathbf{K}_r = \langle \delta | \tag{7.10.b}$$

- perturbation control

$$u_p = -\mathbf{K}_p.|\mathbf{w}|.\operatorname{sgn}(s\mathbf{H}\overline{\mathbf{g}}), \quad \mathbf{K}_p = \left\langle \frac{\sup_i - \inf_i}{2} \right| \tag{7.10.c}$$

with $\mathbf{w}, \sup_i, \inf_i$ *defined* by

$$-u_{eq} = (\mathbf{H}\tilde{\mathbf{g}})^{-1}\mathbf{H}\tilde{\mathbf{f}} = \left\langle \begin{matrix}\mathbf{Sup}\\ \mathbf{Inf}\end{matrix} \right\rangle.\mathbf{w} = \sum_i \left\langle \begin{matrix}\sup_i\\ \inf_i\end{matrix} \right\rangle.w_i, \quad \left\langle \begin{matrix}\mathbf{Sup}\\ \mathbf{Inf}\end{matrix} \right\rangle = \left\langle \begin{matrix}\sup_i\\ \inf_i\end{matrix} \right\rangle, \quad \mathbf{w} = \mathbf{w}(\mathbf{x}) = [w_i(\mathbf{x})]$$

where

$$\mathbf{x}, \tilde{\mathbf{f}}, \tilde{\mathbf{g}} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re, \quad \delta \in \Re_{(+)} : \text{sliding margin.}$$

Proof

Consider a hyperplane

$$s = \mathbf{H}\mathbf{x} \Rightarrow \dot{s} = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}\left(\tilde{\mathbf{f}} + \tilde{\mathbf{g}}.u\right) = \mathbf{H}\tilde{\mathbf{g}}.\left[u + (\mathbf{H}\tilde{\mathbf{g}})^{-1}\mathbf{H}\tilde{\mathbf{f}}\right] = \mathbf{H}\tilde{\mathbf{g}}.\left(u - u_{eq}\right)$$

then

$$-u_{eq} = (\mathbf{H}\tilde{\mathbf{g}})^{-1}\mathbf{H}\tilde{\mathbf{f}} = \sum \left\langle \begin{matrix}\sup_i\\ \inf_i\end{matrix} \right\rangle \times w_i(\mathbf{x})$$

Let

$$u = \sum k_i \times w_i(\mathbf{x})$$

then, to satisfy the sliding condition

$$s\dot{s} < 0 \Rightarrow k_i \begin{cases} < -\sup_i, & \text{if } \; s\mathbf{H}\tilde{\mathbf{g}}w_i > 0 \\ > -\inf_i, & \text{if } \; s\mathbf{H}\tilde{\mathbf{g}}w_i < 0 \end{cases} \Rightarrow k_i = \begin{cases} -\sup_i - \delta, & \text{if } \; s\mathbf{H}\tilde{\mathbf{g}}w_i > 0 \\ -\inf_i + \delta, & \text{if } \; s\mathbf{H}\tilde{\mathbf{g}}w_i < 0 \end{cases}$$

thus, by Lemma 3.1, under Assumption 7.1, the discontinuous SMC control function is

$$u = -\sum \left(\frac{\sup_i + \inf_i}{2} \right) \times w_i(\mathbf{x}) - \sum \left(\delta + \frac{\sup_i - \inf_i}{2} \right) \times |w_i(\mathbf{x})|.\operatorname{sgn}(s\mathbf{H}\overline{\mathbf{g}})$$

Q.E.D.

Remark 7.3: Nonlinearity as Uncertainty

We can apply this robust control formula for the nonlinear case where the nonlinearity is considered as an uncertainty within an operating range.

Remark 7.4: Alleviation of Chattering

To eliminate the chattering problem, for simplicity, the discontinuous SMC control functions in Eqs.(7.7) and (7.8) are smoothed out by the following substitution of a hyperbolic tangent function (Section 3.5.4)

$$\text{sgn}(s) \rightarrow \tanh(k_s s) \tag{7.11}$$

where k_s is chosen low enough to eliminate the chattering problem. This is a continuous pseudo-SMC (Remark 3.4)

Remark 7.5: Performance of Pseudo-SMC

As stated in Remark 4.14, there is negligible difference between a true-SMC and a pseudo-SMC.

7.4. ROBUST CONTINUOUS SMC DESIGN FOR SISO UNCERTAIN NONLINEAR SYSTEMS

In this section, we will consider a SMC design for SISO nonlinear systems, then a robust pseudo-SMC design for SISO uncertain nonlinear systems.

7.4.1. Continuous SMC Design for SISO Nonlinear Systems

Under no perturbation, we have the following theorem.

Theorem 7.4: Continuous SMC Design for Nonlinear Systems

Consider a canonical nonlinear SISO system

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t).u$$

then a continuous SMC control function is determined by

$$\underline{\underline{u = u_e + u_r}} \tag{7.12}$$

where

- equivalent control

$$u_e = -(\mathbf{Hg})^{-1}\mathbf{Hf}, \tag{7.12.a}$$

- reaching control

$$u_r = -(\mathbf{Hg})^{-1}.\delta.s \tag{7.12.b}$$

where

$$\mathbf{x}, \mathbf{f}, \mathbf{g} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re, \quad \delta \in \Re_{(+)} \text{ : sliding margin.}$$

Proof

Since the canonical form, the direct eigenvalue allocation is applied for a hyperplane

$$s = \mathbf{Hx} \Rightarrow \dot{s} = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}(\mathbf{f} + \mathbf{g}.u) = \mathbf{Hg}.\left[u + (\mathbf{Hg})^{-1}\mathbf{Hf}\right]$$

by the above continuous SMC control function, we have

$$s\dot{s} = -\delta.s^2 < 0 \text{: the sliding condition is satisfied.}$$

Q.E.D.

7.4.2. Robust Continuous Pseudo-SMC Design for SISO Uncertain Nonlinear Systems

For an uncertain nonlinear system, we have only a pseudo-SMC which is determined by the following corollary

Corollary 7.2: Robust Continuous Pseudo-SMC Design for Uncertain Nonlinear Systems

Consider a canonical uncertain dynamical nonlinear SISO system

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{\mathbf{g}}(\mathbf{x},t).u$$

then, under Assumption 7.1, a discontinuous SMC control function is determined by

$$\underline{\underline{u = u_e + u_r + u_p}} \tag{7.13}$$

where

- equivalent control

$$u_e = -\mathbf{K}_e.\mathbf{w}, \qquad \mathbf{K}_e = \left\langle \frac{\sup_i + \inf_i}{2} \right| \tag{7.13.a}$$

- reaching control

$$u_r = -\mathbf{K}_r.|\mathbf{w}|.\tanh(k_s s \mathbf{H}\overline{\mathbf{g}}), \qquad \mathbf{K}_r = \langle \delta | \tag{7.13.b}$$

- perturbation control

$$u_p = -\mathbf{K}_p.|\mathbf{w}|.\tanh(k_s s \mathbf{H}\overline{\mathbf{g}}), \qquad \mathbf{K}_p = \left\langle \frac{\sup_i - \inf_i}{2} \right| \tag{7.13.c}$$

with $\mathbf{w}, \sup_i, \inf_i$ *defined* by

$$-u_{eq} = (\mathbf{H}\tilde{\mathbf{g}})^{-1}\mathbf{H}\tilde{\mathbf{f}} = \left\langle \begin{matrix}\mathbf{Sup}\\ \mathbf{Inf}\end{matrix} \right\rangle.\mathbf{w} = \sum_i \left\langle \begin{matrix}\sup_i\\ \inf_i\end{matrix} \right\rangle.w_i, \qquad \left\langle \begin{matrix}\mathbf{Sup}\\ \mathbf{Inf}\end{matrix} \right\rangle = \left\langle \begin{matrix}\sup_i\\ \inf_i\end{matrix} \right\rangle, \qquad \mathbf{w} = \mathbf{w}(\mathbf{x}) = [w_i(\mathbf{x})]$$

where

$$\mathbf{x}, \tilde{\mathbf{f}}, \tilde{\mathbf{g}} \in \Re^{n\times 1}, \quad \mathbf{H} \in \Re^{1\times n}, \quad u, s \in \Re, \quad \delta \in \Re_{(+)} : \text{sliding margin.}$$

Proof

By Section 3.5.4, the sign function is replaced by the hyperbolic tangent function to eliminate the chattering problem.

Q.E.D.

7.5. GENERAL SMC DESIGN FOR SISO NONLINEAR SYSTEMS

For a general case, now the output does not have to be the first state, $y \neq x_1$, it can be a possibly nonlinear function of system states, $y = \eta(\mathbf{x})$. First we have a discontinuous SMC, then a continuous SMC. For a discontinuous SMC, we have the following theorem

Theorem 7.5: General Discontinuous SMC Design for Nonlinear Systems

Consider a general nonlinear SISO system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t).u \\ y = \eta(\mathbf{x}) \end{cases} \tag{7.14}$$

with the relative degree of n, then a *discontinuous* control function can be found by

$$\underline{\underline{u = u_e + u_r}} \tag{7.15}$$

where

- equivalent control
$$u_e = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(L_{\mathbf{f}}^{n}\eta + \mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)}\right), \tag{7.15.a}$$

- reaching control
$$u_r = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}.\delta.\mathrm{sgn}(s), \quad s = \mathbf{H}.\mathbf{z} \tag{7.15.b}$$

with
$$\mathbf{x},\mathbf{f},\mathbf{g} \in \Re^{n\times 1}, \quad u,y \in \Re, \quad \delta \in \Re_{(+)}, \quad \mathbf{z} = \begin{bmatrix} \eta(\mathbf{x}) & \dfrac{d\eta(\mathbf{x})}{dt} & \cdots & \dfrac{d^{n-2}\eta(\mathbf{x})}{dt^{n-2}} & \dfrac{d^{n-1}\eta(\mathbf{x})}{dt^{n-1}} \end{bmatrix}^T;$$
$$\mathbf{H}_{(1:n-1)} = \begin{bmatrix} h_1, & h_2, & \cdots, & h_{n-2}, & h_{n-1} \end{bmatrix}, \qquad \mathbf{z}_{(2:n)} = \begin{bmatrix} z_2, & z_3, & \cdots, & z_{n-1}, & z_n \end{bmatrix}^T$$

Proof

By the I/O state method, keep differentiating y until u first appears, since the relative degree r, under the *Lie derivative*'s notation:
$$\begin{cases} \dot{y} = L_{\mathbf{f}}\eta \\ \ddot{y} = L_{\mathbf{f}}^2\eta \\ \cdots \\ y^{(n-1)} = L_{\mathbf{f}}^{n-1}\eta \\ y^{(n)} = L_{\mathbf{f}}^{n}\eta + L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta.u, \quad \text{with} \quad L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta \neq 0 \end{cases}$$

thus
$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix} \Rightarrow \dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{bmatrix}$$

then
$$\dot{\mathbf{z}} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_r \\ v \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_r \\ L_{\mathbf{f}}^{n}\eta + L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta.u \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_r \\ L_{\mathbf{f}}^{n}\eta \end{bmatrix} + \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_r \\ L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta \end{bmatrix}.u$$

where
$$v = L_{\mathbf{f}}^{n}\eta + L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta.u \Leftrightarrow u = \left(L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(v - L_{\mathbf{f}}^{n}\eta\right)$$

then by the direct allocation method, we have a hyperplane
$$s = \mathbf{H}\mathbf{z} = \begin{bmatrix} h_1, & \cdots, & h_{n-1}, & 1 \end{bmatrix}.\mathbf{z} \Rightarrow \dot{s} = \mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)} + v = v - v_{eq}$$

with
$$v_{eq} = -\mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)}$$

To satisfy the sliding condition,
$$s.\dot{s} < 0 \Rightarrow s\left(v - v_{eq}\right) < 0$$

then the control function is

$$v = v_{eq} - \delta.\text{sgn}(s) \Rightarrow s.\dot{s} = -\delta.|s| < 0$$

thus

$$u = \left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(v_{eq} - \delta.\text{sgn}(s) - L_{\mathbf{f}}^{n}\eta\right) = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(\mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)} + \delta.\text{sgn}(s) + L_{\mathbf{f}}^{n}\eta\right)$$

Q.E.D.

Now for a continuous SMC, we have the following theorem

Theorem 7.6: General Continuous SMC Design for Nonlinear Systems

Consider a general nonlinear SISO system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t).u; \qquad y = \eta(\mathbf{x})$$

with the relative degree of n, then a *continuous* control function can be found by

$$\underline{\underline{u = u_e + u_r}} \tag{7.16}$$

where

- equivalent control

$$u_e = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(L_{\mathbf{f}}^{n}\eta + \mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)}\right) \tag{7.16.a}$$

- reaching control

$$u_r = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}.\delta.s, \qquad s = \mathbf{H}.\mathbf{z} \tag{7.16.b}$$

where

$$\mathbf{x},\mathbf{f},\mathbf{g} \in \Re^{n\times 1}, \quad u,y \in \Re, \quad \delta \in \Re_{(+)} \qquad \mathbf{z} = \begin{bmatrix} \eta(\mathbf{x}) & \dfrac{d\eta(\mathbf{x})}{dt} & \cdots & \dfrac{d^{n-2}\eta(\mathbf{x})}{dt^{n-2}} & \dfrac{d^{n-1}\eta(\mathbf{x})}{dt^{n-1}} \end{bmatrix}^T$$

$$\mathbf{H}_{(1:n-1)} = \left[h_1, \quad h_2, \quad \cdots, \quad h_{n-2}, \quad h_{n-1}\right], \qquad \mathbf{z}_{(2:n)} = \left[z_2, \quad z_3, \quad \cdots, \quad z_{n-1}, \quad z_n\right]^T.$$

Proof

Similar to the above proof, but now for a continuous SMC, to satisfy the sliding condition, the control function is

$$v = v_{eq} - \delta.s \Rightarrow s.\dot{s} = -\delta.s^2 < 0$$

thus

$$u = \left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(v_{eq} - \delta.s - L_{\mathbf{f}}^{n}\eta\right) = -\left(L_{\mathbf{g}} L_{\mathbf{f}}^{n-1}\eta\right)^{-1}\left(\mathbf{H}_{(1:n-1)}\mathbf{z}_{(2:n)} + \delta.s + L_{\mathbf{f}}^{n}\eta\right).$$

Q.E.D.

Part B : General SMC Design for MIMO Uncertain Nonlinear Systems

In this part, we will use a simple decoupling method to convert a MIMO system into a multiple SISO subsystem, then all SISO results developed so far can be applied, including robust results and even the recent nonlinear result.

7.6. GENERAL SMC DESIGN FOR MIMO UNCERTAIN NONLINEAR SYSTEMS

This section will generalize the SISO technique above for MIMO nonlinear

Theorem 7.7: General Discontinuous SMC Design for MIMO Nonlinear Systems

Consider a general nonlinear MIMO system

$$\left.\begin{array}{l}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}).\mathbf{u};\\ \mathbf{y} = \eta(\mathbf{x});\end{array}\right\}, \quad \left.\begin{array}{l}\mathbf{x} \in \mathbf{R}^n;\\ \mathbf{u},\mathbf{y} \in \mathbf{R}^m;\end{array}\right\}, \quad \mathbf{G}(\mathbf{x}) = \left[\mathbf{g}_1(\mathbf{x}) \quad \cdots \quad \mathbf{g}_m(\mathbf{x})\right], \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{7.17}$$

If the system Jacobian $L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta(\mathbf{x})$ is non-singular, then a discontinuous control function can be found from

$$\underline{\underline{\mathbf{u} = \mathbf{u}_e + \mathbf{u}_r}} \tag{7.18}$$

where

- equivalent control

$$\mathbf{u}_e = -\left(L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r-1}} \eta\right)^{-1} \left(L_{\mathbf{f}}^{\mathbf{r}} \eta + \mathbf{H}_{(\mathbf{1:r-1})} \mathbf{z}_{(2:n)}\right), \tag{7.18.a}$$

- reaching control

$$\mathbf{u}_r = -\left(L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r-1}} \eta\right)^{-1} .\delta.\mathrm{sgn}(\mathbf{s}), \qquad \mathbf{s} = \mathbf{H}.\mathbf{z} \tag{7.18.b}$$

Proof

Similarly to the case of SISO

$$\begin{cases} \dot{y}_1 = L_{\mathbf{f}} \eta_1 \\ \ddot{y}_1 = L_{\mathbf{f}}^2 \eta_1 \\ \cdots \\ y_1^{(r_1-1)} = L_{\mathbf{f}}^{r_1-1} \eta_1 \\ y_1^{(r_1)} = L_{\mathbf{f}}^{r_1} \eta_1 + L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_1-1} \eta_1 .u \end{cases} \qquad \cdots \qquad \begin{cases} \dot{y}_m = L_{\mathbf{f}} \eta_m \\ \ddot{y}_m = L_{\mathbf{f}}^2 \eta_m \\ \cdots \\ y_m^{(r_m-1)} = L_{\mathbf{f}}^{r_m-1} \eta_m \\ y_m^{(r_m)} = L_{\mathbf{f}}^{r_m} \eta_1 + L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_m-1} \eta_m .u \end{cases}$$

where

$$\mathbf{r} = \left[r_1 \quad \cdots \quad r_m\right] : \textit{ relative degree} \text{ of the system}$$

then

$$\mathbf{y}^{(\mathbf{r})} = \mathbf{v}, \qquad \mathbf{v} = L_{\mathbf{f}}^{\mathbf{r}} \eta + L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta.\mathbf{u}$$

with

$$L_{\mathbf{f}}^{\mathbf{r}} \eta = \begin{bmatrix} L_{\mathbf{f}}^{r_1} \eta_1 \\ L_{\mathbf{f}}^{r_2} \eta_2 \\ \vdots \\ L_{\mathbf{f}}^{r_m} \eta_m \end{bmatrix}, \qquad L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{G}} L_{\mathbf{f}}^{r_1-1} h_1(\mathbf{x}) \\ L_{\mathbf{G}} L_{\mathbf{f}}^{r_2-1} h_1(\mathbf{x}) \\ \vdots \\ L_{\mathbf{G}} L_{\mathbf{f}}^{r_m-1} h_1(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_1-1} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_1-1} h_1(\mathbf{x}) \\ L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_2-1} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_2-1} h_1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_m-1} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_m-1} h_1(\mathbf{x}) \end{bmatrix}$$

since the system Jacobian $L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta(\mathbf{x})$ is non-singular, so

$$\mathbf{u} = \left(L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta\right)^{-1} \left(\mathbf{v} - L_{\mathbf{f}}^{\mathbf{r}} \eta\right)$$

thus the control function, elementwisely

$$\begin{cases} u_i = -\left(L_{\mathbf{G}} L_{\mathbf{f}}^{r_i-1} \eta\right)^{-1} \left[L_{\mathbf{f}}^{r_i} \eta + \mathbf{H}_{(1-r_i)}^{(i)} \mathbf{z}_{(2:n)}^{(i)} + \delta^{(i)} . \mathrm{sgn}(s_i) \right] \\ s_i = \mathbf{H}^{(i)} . \mathbf{z}^{(i)} \end{cases}$$

Q.E.D.

For a continuous SMC control function

Theorem 7.8: General Continuous SMC Design for MIMO Nonlinear Systems

Consider a general nonlinear MIMO system

$$\left.\begin{matrix} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \\ \mathbf{y} = \eta(\mathbf{x}) \end{matrix}\right\}, \quad \left.\begin{matrix} \mathbf{x} \in \mathbf{R}^n \\ \mathbf{u}, \mathbf{y} \in \mathbf{R}^m \end{matrix}\right\}, \quad \mathbf{G}(\mathbf{x}) = \left[\mathbf{g}_1(\mathbf{x}) \quad \cdots \quad \mathbf{g}_m(\mathbf{x})\right], \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

If the system Jacobian $L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta(\mathbf{x})$ is non-singular, then the continuous control function is determined by

$$\underline{\underline{\mathbf{u} = \mathbf{u}_e + \mathbf{u}_r}} \tag{7.19}$$

where

- equivalent control

$$\mathbf{u}_e = -\left(L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta\right)^{-1} \left(L_{\mathbf{f}}^{\mathbf{r}} \eta + \mathbf{H}_{(\mathbf{1}:\mathbf{r}-\mathbf{1})} \mathbf{z}_{(2:n)}\right), \tag{7.19.a}$$

- reaching control

$$\mathbf{u}_r = -\left(L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta\right)^{-1} . \delta . \mathbf{s}, \qquad \mathbf{s} = \mathbf{H} . \mathbf{z} \tag{7.19.b}$$

Proof.

Similarly to the above proof of Theorem 7.7.

Q.E.D.

Remark 7.6: Singular System Jacobian

If the system Jacobian $L_{\mathbf{G}} L_{\mathbf{f}}^{\mathbf{r}-1} \eta(\mathbf{x})$ is singular, then the Structure Algorithm (Silverman 1969) and the error dynamic equation (Chen *et al.* 1992) must be used.

Remark 7.7: A Robust SMC Design for General MIMO Uncertain Nonlinear Systems

For a class of MIMO systems whose system Jacobians are non-singular, we can use a simple decoupling technique to get a multiple SISO subsystem, then all the robust SISO results so far are directly applicable for a robust general SMC design for MIMO uncertain nonlinear systems. It is a very direct extension, Example 7.6 illustrates this issue.

7.7. NUMERICAL EXAMPLES.

Because it is quite feasible for a SMC to deal with nonlinear systems, so there is no differences, and hence no difficulties in the design approach between the *regulating control* and the *tracking control*. Thus it is illustrated by an example without any special development.

This section is crucial in the fact that it clarifies something left open previously for the sake of simplicity, such as the design of a tracking control, the definition of **w** in an equivalent control (Section 7.3), the decoupling technique to convert a MIMO system into a multiple SISO subsystem, a robust MIMO general SMC controller design (Remark 7.7).

There is no original design available for all the following examples, the proposed designs are based on the *design guideline* in Proposition 3.1.

For MIMO examples, Example 7.4 to 7.6, each has 2 simulations to check interactions after decoupling, only the first output is under control in the first simulation, then both are under control in the second simulation.

Remark 7.8: Summary of SMC for Uncertain MIMO Systems

We will consider only canonical nonlinear system

- Hyperplane for a SISO canonical nonlinear system can be computed using Theorem 7.1;
- Continuous SMC for SISO nonlinear systems can be designed using Theorem 7.4. There is no such design for uncertain systems;
- Discontinuous SMC for SISO uncertain nonlinear systems can be designed using Theorem 7.3 where deterministic systems are special cases (or Theorem 7.2 can be attempted);
- For general SISO nonlinear systems where the output is a function of system states, discontinuous and continuous SMC can be designed using Theorem 7.5 and 7.6, respectively; while Theorem 7.7 and 7.8 for MIMO systems.

7.7.1. Example 7.1: SISO Nonlinear System

Consider a system in Zhou *et al.* 1992:

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}.u \\ y = x_1\end{cases}, \qquad \mathbf{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1)\end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 1+\sqrt{|x_1|}\end{bmatrix}$$

choose a hyperplane-eigenvalue

$$\lambda_H = [-2] \Rightarrow \mathbf{H} = [2 \quad 1]$$

By Eq.(7.7), the equivalent control is

$$-u_{eq} = (\mathbf{Hg})^{-1}\mathbf{Hf} = \left(1+\sqrt{|x_1|}\right)^{-1}\left[2x_2 + 2x_1x_2 + x_1^2 + \sin(tx_1)\right]$$

thus $\mathbf{K}_e = \left(1+\sqrt{|x_1|}\right)^{-1} \times [2, \quad 2, \quad 1, \quad 1], \qquad \mathbf{w} = [x_2, \quad x_1x_2, \quad x_1^2, \quad \sin(tx_1)]$

7.7.1.1. Discontinuous SMC

Theorem 7.3 yields a discontinuous SMC as

$$u = -\mathbf{K}_e.\mathbf{w} - \mathbf{K}_r.|\mathbf{w}|.\mathrm{sgn}(s)$$

choose $\delta = 2$ to have

$$\mathbf{K}_r = \left(1 + \sqrt{|x_1|}\right)^{-1} \times [2, \quad 2, \quad 2, \quad 2]$$

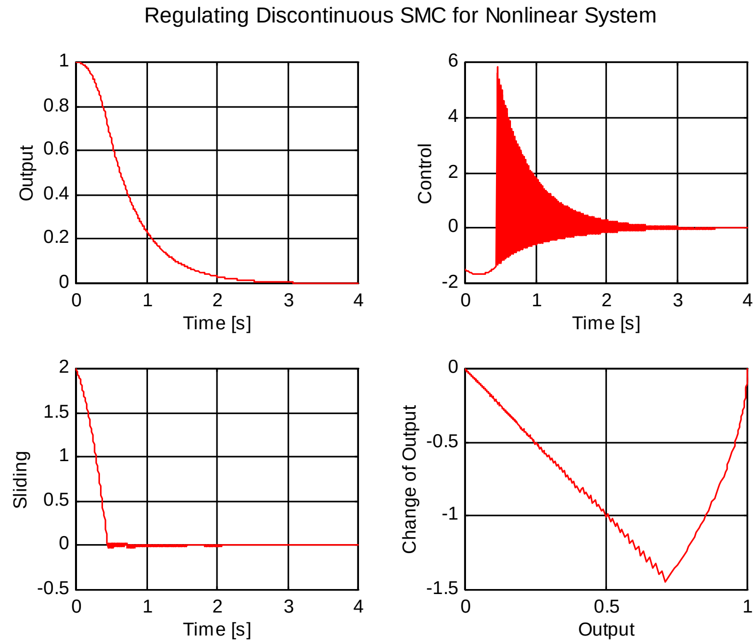


Fig. 7.1: Regulating Discontinuous Nonlinear SMC for Example 7.1.

7.7.1.2. Continuous Pseudo-SMC

In the discontinuous control function above, let $\mathrm{sgn}(s) \to \tanh(k_s s)$ and choose $k_s = 15$, then we have

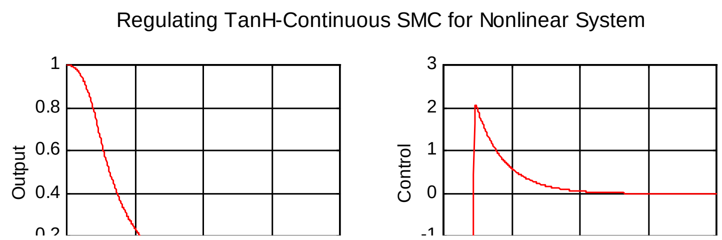


Fig. 7.2: Regulating TanH-Continuous Nonlinear SMC for Example 7.1.

7.7.1.3. Continuous SMC

Theorem 7.4 yields a continuous control function as

$$u = -(\mathbf{Hg})^{-1}[\mathbf{Hf} + \delta.s] = -\left(1+\sqrt{|x_1|}\right)^{-1}\left\{[2, \quad 1].\begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1) \end{bmatrix} + \delta.s\right\}$$

so

$$u = -\left(1+\sqrt{|x_1|}\right)^{-1}\left\{2x_2 + 2x_1x_2 + x_1^2 + \sin(tx_1) + \delta.s\right\}$$

choose $\delta = 4$, then we have

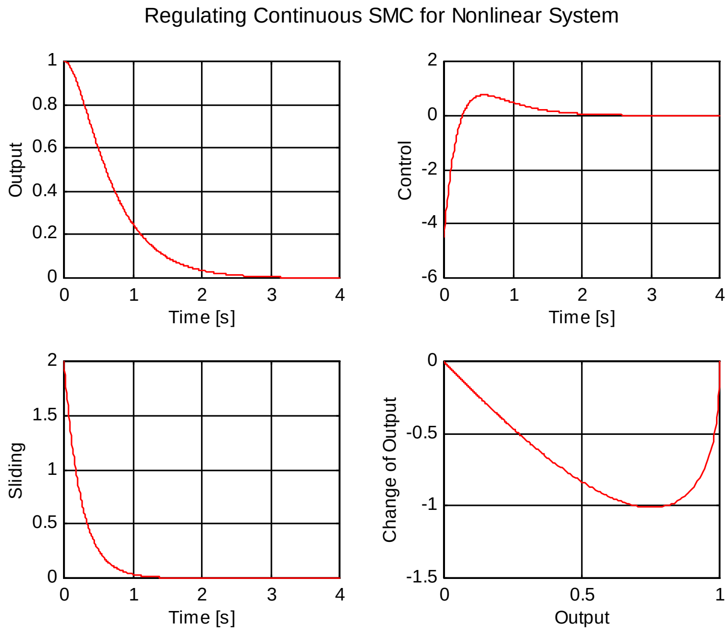


Fig. 7.3: Regulating Continuous Nonlinear SMC of Form 3 for Example 7.1.

7.7.1.4. Performance

The performance of the continuous SMC is comparable to that of the continuous pseudo-SMC.

7.7.2. Example 7.2: SISO Uncertain Nonlinear System

Consider a system in Zhou *et al.* 1992 which is modified with uncertainties

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ ax_1x_2 + bx_1^2 + b\sin(tx_1) + \left(1+\sqrt{|x_1|}\right)u \end{bmatrix}, \qquad a = \left\langle \begin{matrix} 3 \\ 1 \end{matrix} \right\rangle, \qquad b = \left\langle \begin{matrix} 1.5 \\ 0.5 \end{matrix} \right\rangle$$

choose a hyperplane-eigenvalue

$$\lambda_H = [-2] \Rightarrow \mathbf{H} = [2 \quad 1]$$

then the hyperplane

$$s = \mathbf{H}.\mathbf{x}$$

and

$$\dot{s} = \mathbf{H}.\dot{\mathbf{x}} = 2x_2 + ax_1x_2 + bx_1^2 + b\sin(tx_1) + \left(1+\sqrt{|x_1|}\right)u$$

or

$$\dot{s} = \left(1+\sqrt{|x_1|}\right)\left\{u + \left(1+\sqrt{|x_1|}\right)^{-1}\left[2x_2 + ax_1x_2 + bx_1^2 + b\sin(tx_1)\right]\right\}$$

so

$$-u_{eq} = \left(1+\sqrt{|x_1|}\right)^{-1}\left[2x_2 + ax_1x_2 + bx_1^2 + b\sin(tx_1)\right] = \left(1+\sqrt{|x_1|}\right)^{-1}\left[\left\langle \begin{matrix} 2 \\ 2 \end{matrix} \right\rangle x_2 + \left\langle \begin{matrix} 3 \\ 1 \end{matrix} \right\rangle x_1x_2 + \left\langle \begin{matrix} 1.5 \\ 0.5 \end{matrix} \right\rangle x_1^2 + \left\langle \begin{matrix} 1.5 \\ -1.5 \end{matrix} \right\rangle\right]$$

thus

$$\mathbf{w} = \left[1, \quad x_2, \quad x_1x_2, \quad x_1^2\right]$$

7.7.2.1. Robust Discontinuous SMC

Theorem 7.3 yields a control function as

$$u = -\mathbf{K}_e.\mathbf{w} - \mathbf{K}_{rp}.|\mathbf{w}|.\mathrm{sgn}(s)$$

where

$$\mathbf{K}_e = \left(1+\sqrt{|x_1|}\right)^{-1}\left[0, \quad 2, \quad 2, \quad 1\right]$$

and choose $\delta = 0.2$, we have

$$\mathbf{K}_{rp} = \mathbf{K}_r + \mathbf{K}_p = \left(1+\sqrt{|x_1|}\right)^{-1}\left[1.7, \quad 0.2, \quad 1.2, \quad 0.7\right]$$

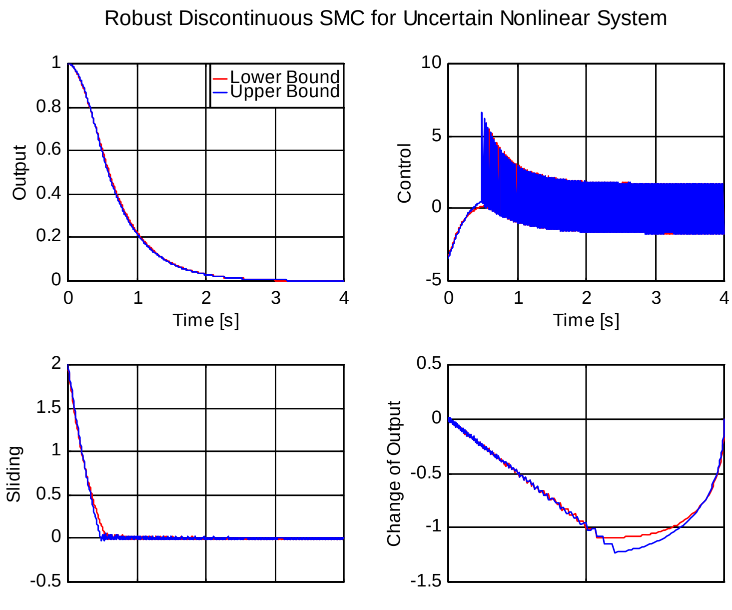


Fig. 7.4: Robust Discontinuous Nonlinear SMC for Example 7.2, 2 curves coincide.

7.7.2.2. Robust Continuous Pseudo-SMC

In the discontinuous control function above, let $\operatorname{sgn}(s) \to \tanh(k_s s)$ and choose $k_s = 30$, then we have

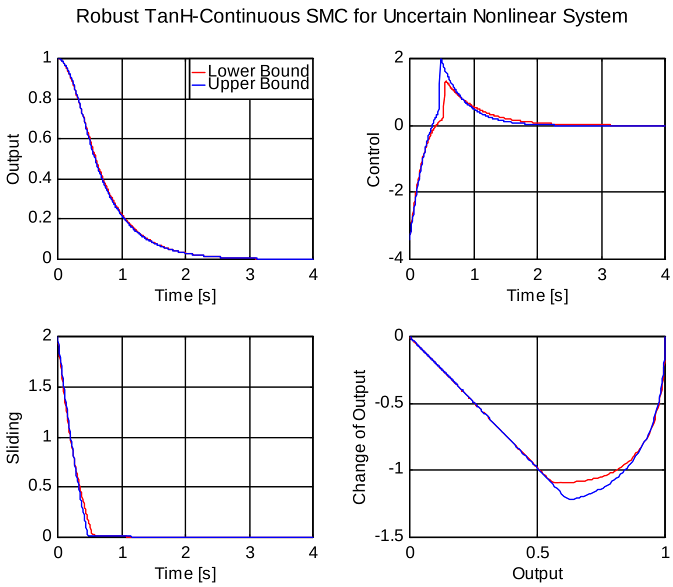


Fig. 7.5: Robust TanH-Continuous Nonlinear SMC for Example 7.2, 2 curves coincide.

7.7.3. Example 7.3: Tracking SISO Nonlinear System

Consider a system in Zhou *et al.* 1992 which is modified with a reference output:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1) + \left(1+\sqrt{|x_1|}\right)u \end{bmatrix} \\ y_{ref} = \ln(t+1) \Rightarrow z = y - y_{ref} = x_1 - \ln(t+1) \end{cases}$$

then

$$\begin{cases} z = x_1 - \ln(t+1) \\ \dot{z} = x_2 - \dfrac{1}{t+1} \\ \ddot{z} = 2x_1x_2 + x_1^2 + \sin(tx_1) + \dfrac{1}{(t+1)^2} + \left(1+\sqrt{|x_1|}\right)u \end{cases}$$

choose a hyperplane-eigenvalue

$$\lambda_H = [-2] \Rightarrow \mathbf{H} = [2 \quad 1]$$

then the hyperplane

$$s = \mathbf{H}.\mathbf{z}$$

so

$$\dot{s} = 2\dot{z} + \ddot{z} = 2\left(x_2 - \frac{1}{t+1}\right) + 2x_1x_2 + x_1^2 + \sin(tx_1) + \frac{1}{(t+1)^2} + \left(1+\sqrt{|x_1|}\right)u$$

or

$$\dot{s} = \left(1+\sqrt{|x_1|}\right)\left\{u + \left(1+\sqrt{|x_1|}\right)^{-1}\left[2\left(x_2 - \frac{1}{t+1}\right) + 2x_1x_2 + x_1^2 + \sin(tx_1) + \frac{1}{(t+1)^2}\right]\right\}$$

hence

$$u_{eq}=-\left(1+\sqrt{|x_1|}\right)^{-1}\left[2\left(x_2-\frac{1}{t+1}\right)+2x_1x_2+x_1^2+\sin(tx_1)+\frac{1}{(t+1)^2}\right]$$

7.7.3.1. Discontinuous SMC

By Theorem 7.3, a discontinuous SMC is determined by

$$u=-\left(1+\sqrt{|x_1|}\right)^{-1}\left\{2\left(x_2-\frac{1}{t+1}\right)+2x_1x_2+x_1^2+\sin(tx_1)+\frac{1}{(t+1)^2}+\delta.\mathrm{sgn}(s)\right\}$$

choose $\delta=2$, we have

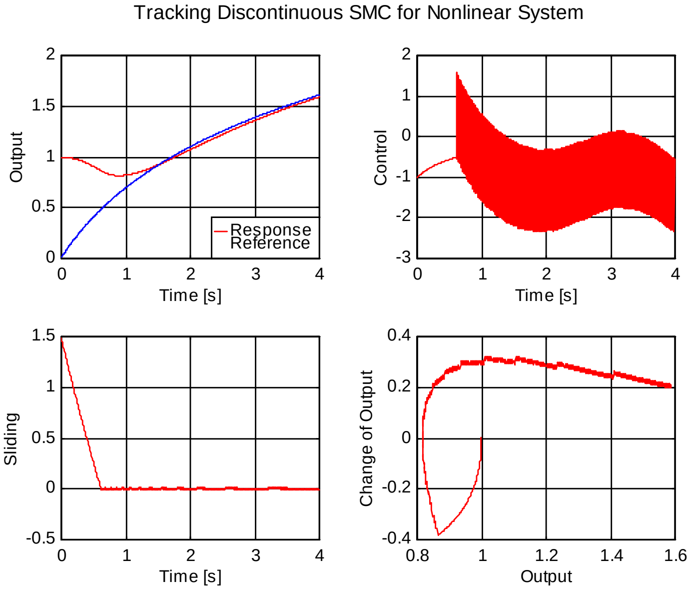


Fig. 7.6: Tracking Discontinuous Nonlinear SMC for Example 7.3.

7.7.3.2. Continuous Pseudo-SMC

In the discontinuous control function above, let $\mathrm{sgn}(s)\rightarrow\tanh(k_s s)$ and choose $k_s=60$, then

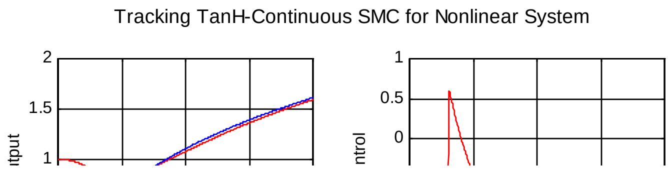


Fig. 7.7: Tracking TanH-Continuous Nonlinear SMC for Example 7.3.

7.7.3.3. Continuous SMC

By Theorem 7.4, a continuous SMC is determined by

$$u=-\left(1+\sqrt{|x_1|}\right)^{-1}\left\{2\left(x_2-\frac{1}{t+1}\right)+2x_1x_2+x_1^2+\sin(tx_1)+\frac{1}{(t+1)^2}+\delta.s\right\}$$

choose $\delta=4$, we have

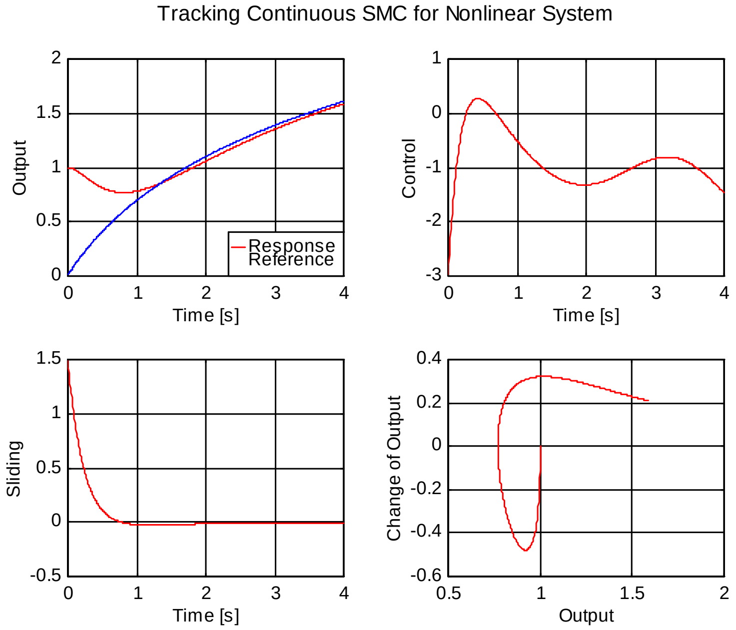


Fig. 7.8: Tracking Continuous Nonlinear SMC for Example 7.3.

7.7.3.4. Performance

The performance of the continuous SMC is comparable to that of the continuous pseudo-SMC.

7.7.4. Example 7.4: MIMO Nonlinear System

Consider a system in Chen *et al.* 1992

$$\begin{cases}\dot{x}_1=x_2+2x_2\left(x_3+u_1+u_2\right)\\ \dot{x}_2=x_3+u_1+u_2\\ \dot{x}_3=x_4-x_3^2-x_5^2\\ \dot{x}_4=x_5+2x_3\left(x_4-x_3^2-x_5^2\right)+2x_5\left(-2x_5-u_2\right)\\ \dot{x}_5=-2x_5-u_2\end{cases},\qquad \begin{cases}y_1=x_1-x_2^2\\ y_2=x_3\end{cases}$$

then

$$\begin{cases}\ddot{y}_1=x_3+u_1+u_2\\ \dddot{y}_2=-2x_5-u_2\end{cases}\Rightarrow\begin{cases}\mathbf{M_g}=L_{\mathbf{G}}L_{\mathbf{f}}^{\mathbf{r}-1}\eta(\mathbf{x})=\begin{bmatrix}1&1\\0&-1\end{bmatrix}\\ \mathbf{M_f}=L_{\mathbf{f}}^{\mathbf{r}}\eta(\mathbf{x})=\begin{bmatrix}x_3\\-2x_5\end{bmatrix}\\ \mathbf{r}=[2,\quad 3]\end{cases}$$

thus

$$\mathbf{u} = \mathbf{M_g}^{-1}\left(\mathbf{v} - \mathbf{M_f}\right)$$

hence

$$\begin{cases} y_{11} = x_1 - x_2^2 \\ y_{12} = x_2 \end{cases} \Rightarrow \begin{cases} \dot{y}_{11} = y_{12} \\ \dot{y}_{12} = v_1 \end{cases}, \qquad \begin{cases} y_{21} = x_3 \\ y_{22} = x_4 - x_3^2 - x_5^2 \\ y_{23} = x_5 \end{cases} \Rightarrow \begin{cases} \dot{y}_{21} = y_{22} \\ \dot{y}_{22} = y_{23} \\ \dot{y}_{23} = v_2 \end{cases}$$

where

$$\begin{bmatrix} \dot{y}_{12} \\ \dot{y}_{23} \end{bmatrix} = \mathbf{M_f} + \mathbf{M_g}\mathbf{u} = \mathbf{M_f} + \mathbf{M_g}\left[\mathbf{M_g}^{-1}\left(\mathbf{v} - \mathbf{M_f}\right)\right] = \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

thus

$$\begin{cases} s_1 = h_{11}y_{21} + y_{12} \\ \dot{s}_1 = h_{11}\dot{y}_{11} + \dot{y}_{12} = h_{11}y_{12} + v_1 \end{cases}, \qquad \begin{cases} s_2 = h_{21}y_{21} + h_{22}y_{22} + y_{23} \\ \dot{s}_2 = h_{21}\dot{y}_{21} + h_{22}\dot{y}_{22} + \dot{y}_{23} = h_{21}y_{22} + h_{22}y_{23} + v_2 \end{cases}$$

choose the hyperplane-eigenvalues as

$$\lambda_{H1} = [-2] \Rightarrow \mathbf{H}_1 = [2 \quad 1], \qquad \lambda_{H2} = [-2 \quad -2] \Rightarrow \mathbf{H}_2 = [4 \quad 4 \quad 1]$$

Therefore, the control function is

$$\mathbf{u} = -\mathbf{M_g}^{-1}\left(\mathbf{M_f} + \mathbf{M_s}\right)$$

with

$$\mathbf{M_g} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{M_f} = \begin{bmatrix} x_3 \\ -2x_5 \end{bmatrix}$$

7.7.4.1. Discontinuous SMC

Theorem 7.7 yields

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{H}_1(1).y_{12} + \delta_1.\mathrm{sgn}(s_1) \\ \mathbf{H}_2(1).y_{22} + \mathbf{H}_2(2).y_{23} + \delta_2.\mathrm{sgn}(s_2) \end{bmatrix}, \qquad \delta_1 = 4, \qquad \delta_2 = 0.2$$

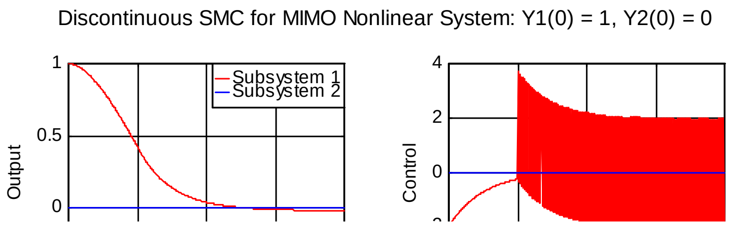


Fig. 7.9: MIMO Discontinuous Nonlinear SMC for Example 7.4: Y1(0) = 1, Y2(0) = 0

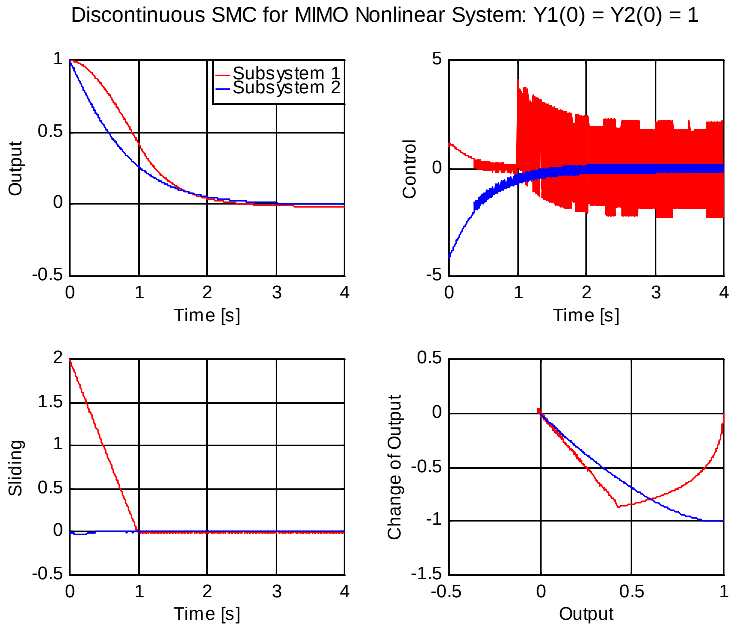


Fig. 7.10: MIMO Discontinuous Nonlinear SMC for Example 7.4: Y1(0) = Y2(0) = 1

7.7.4.2. Continuous Pseudo-SMC

In the above discontinuous functions, let $\operatorname{sgn}(s) \to \tanh(k_s s)$ and choose $k_{s1} = 50,\ k_{s2} = 1000$ to have

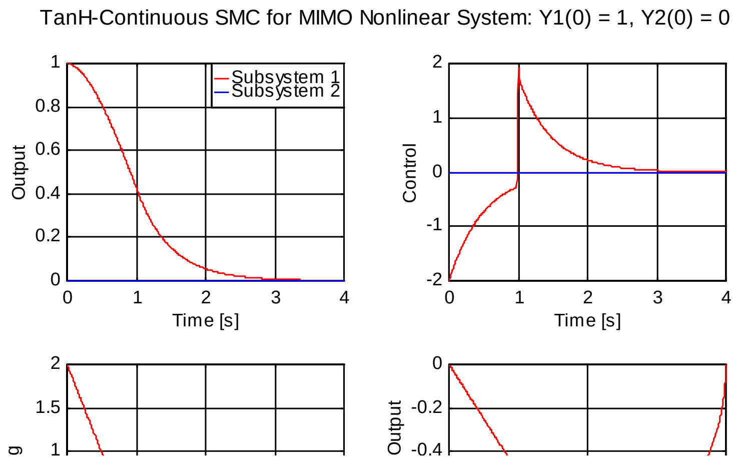


Fig. 7.11: MIMO TanH-Continuous Nonlinear SMC for Example 7.4: Y1(0) = 1, Y2(0) = 0

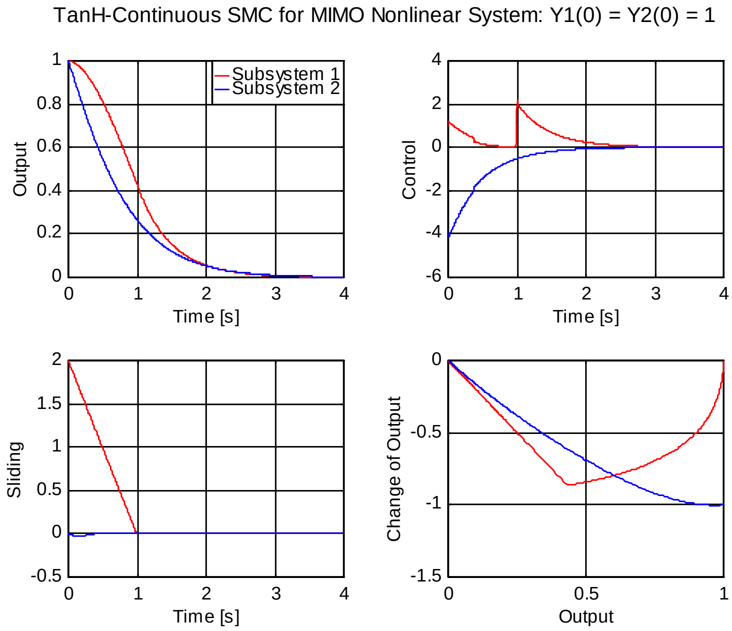


Fig. 7.12: MIMO TanH-Continuous Nonlinear SMC for Example 7.4: Y1(0) = Y2(0) = 1

7.7.4.3. Continuous SMC

Theorem 7.8 yields

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{H}_1(1).y_{12} + \delta_1.s_1 \\ \mathbf{H}_2(1).y_{22} + \mathbf{H}_2(2).y_{23} + \delta_2.s_1 \end{bmatrix}, \qquad \delta_1 = \delta_2 = 4$$

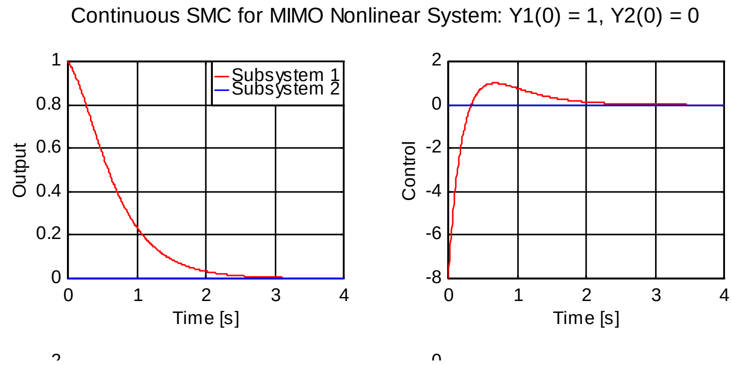


Fig. 7.13: MIMO Continuous Nonlinear SMC for Example 7.4: Y1(0) = 1, Y2(0) = 0

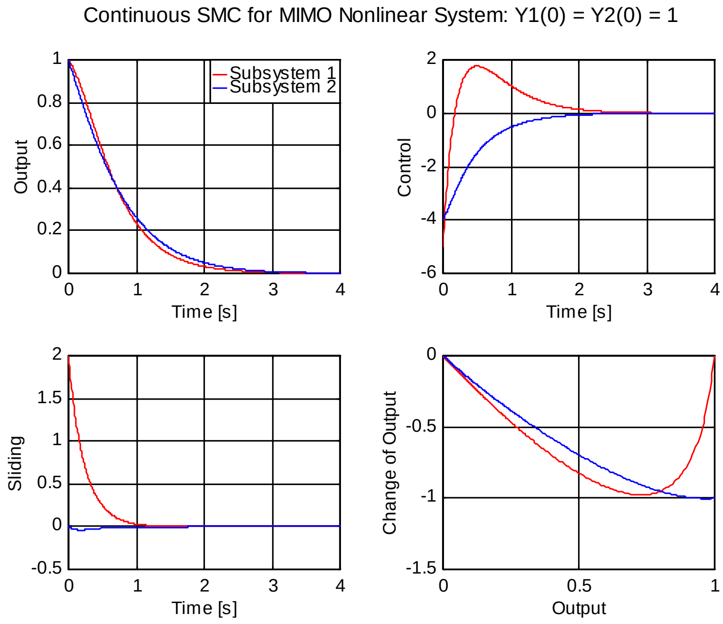


Fig. 7.14: MIMO Continuous Nonlinear SMC for Example 7.4: Y1(0) = Y2(0) = 1

7.7.4.4. Performance

- The decoupling is effective as the second output is almost unaffected. The second output is essentially equal to zero if it is initially at zero and under no control. It goes to zero from an initial value if it is under control as required.
- The performance of the continuous SMC is comparable to that of the continuous pseudo-SMC since for the latter, to eliminate the chattering completely, it is necessary to reduce the gain k_s further, but the steady-state error will be noticeable. This is consistent with Remark 3.5.

7.7.5. Example 7.5: MIMO Nonlinear Robot Manipulators

Consider a robot system in Young 1978

$$\dot{x}_1 = x_2;$$

$$\dot{x}_2 = \frac{\left\{\begin{array}{l} 9\left[5\sin x_3 . x_4\left(2x_2 + x_4\right) - 66.15\cos x_1 - 49\cos\left(x_1 + x_3\right) + u_1\right]\cdots \\ -\left(4 + 5\cos x_3\right)\left[-5\sin x_3 . x_2^2 - 49\cos\left(x_1 + x_3\right) + u_2\right] \end{array}\right\}}{125.75 + 50\cos x_3 - 25\cos^2 x_3};$$

and

$$\dot{x}_3 = x_4;$$

$$\dot{x}_4 = \frac{\left\{\begin{array}{l} -\left(4 + 5\cos x_3\right)\left[5\sin x_3 . x_4\left(2x_2 + x_4\right) - 66.15\cos x_1 - 49\cos\left(x_1 + x_3\right) + u_1\right]\cdots \\ +\left(15.75 + 10\cos x_3\right)\left[-5\sin x_3 . x_2^2 - 49\cos\left(x_1 + x_3\right) + u_2\right] \end{array}\right\}}{125.75 + 50\cos x_3 - 25\cos^2 x_3};$$

then the control function

$$\mathbf{u} = -\mathbf{M}_g^{-1}\left(\mathbf{M}_f + \mathbf{M}_s\right);$$

where

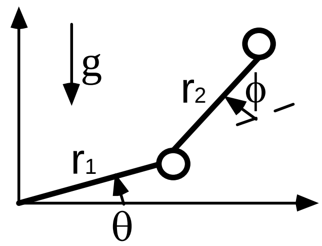
$$\mathbf{M}_g = \left(125.75 + 50\cos x_3 - 25\cos^2 x_3\right)^{-1} \begin{bmatrix} 9 & -\left(4 + 5\cos x_3\right) \\ -\left(4 + 5\cos x_3\right) & 15.75 + 10\cos x_3 \end{bmatrix};$$

and

$$\mathbf{M}_f = \mathbf{M}_g \begin{bmatrix} 5\sin x_3 . x_4\left(2x_2 + x_4\right) - 66.15\cos x_1 - 49\cos\left(x_1 + x_3\right) \\ -5\sin x_3 . x_2^2 - 49\cos\left(x_1 + x_3\right) \end{bmatrix}$$

Choose

$$\left.\begin{matrix} \lambda_{H1} = [-1] \Rightarrow \mathbf{H}_1 = [1 \quad 1] \\ \lambda_{H2} = [-1] \Rightarrow \mathbf{H}_2 = [1 \quad 1] \end{matrix}\right\} \Rightarrow \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 [x_1 \quad x_2]^T \\ \mathbf{H}_2 [x_3 \quad x_4]^T \end{bmatrix}$$



7.7.5.1. Discontinuous SMC

Theorem 7.7 yields

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{H}_1(1). x_2 + \delta_1 .\mathrm{sgn}(s_1) \\ \mathbf{H}_2(1). x_4 + \delta_2 .\mathrm{sgn}(s_2) \end{bmatrix}; \qquad \delta_1 = \delta_2 = 0.5;$$

Discontinuous SMC for MIMO Robotic Nonlinear System: Y1(0) = 1, Y2(0) = 0

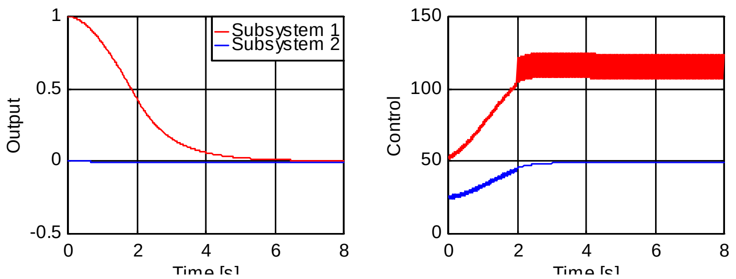


Fig. 7.15: MIMO Discontinuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 0

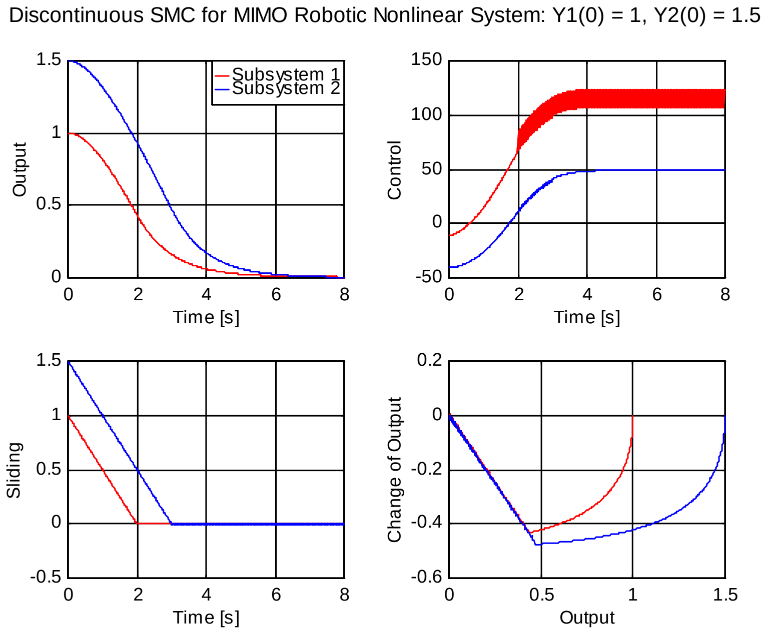


Fig. 7.16: MIMO Discontinuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 1.5

7.7.5.2. Continuous Pseudo-SMC

In the above discontinuous functions, let $\mathrm{sgn}(s) \to \tanh(k_s s)$ and choose $k_{s1} = k_{s2} = 250$ to have

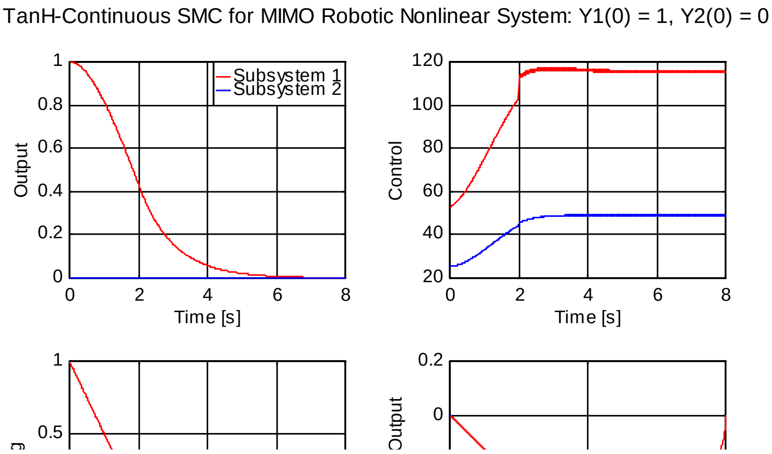


Fig. 7.17: MIMO TanH-Continuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 0

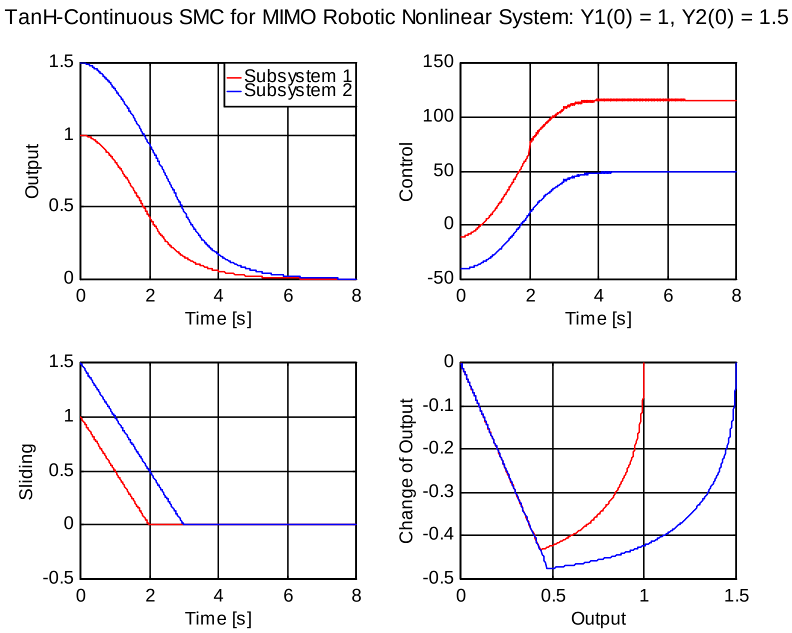


Fig. 7.18: MIMO TanH-Continuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 1.5

7.7.5.3. Continuous SMC

Theorem 7.8 yields

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{H}_1(1).x_2 + \delta_1.s_1 \\ \mathbf{H}_2(1).x_4 + \delta_2.s_2 \end{bmatrix}, \qquad \delta_1 = \delta_2 = 2$$

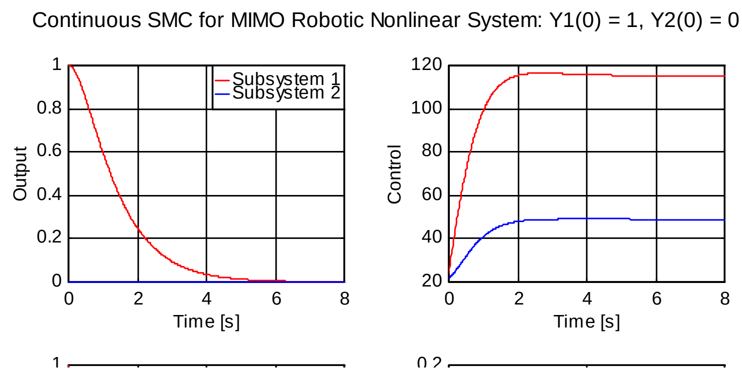


Fig. 7.19: MIMO Continuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 0

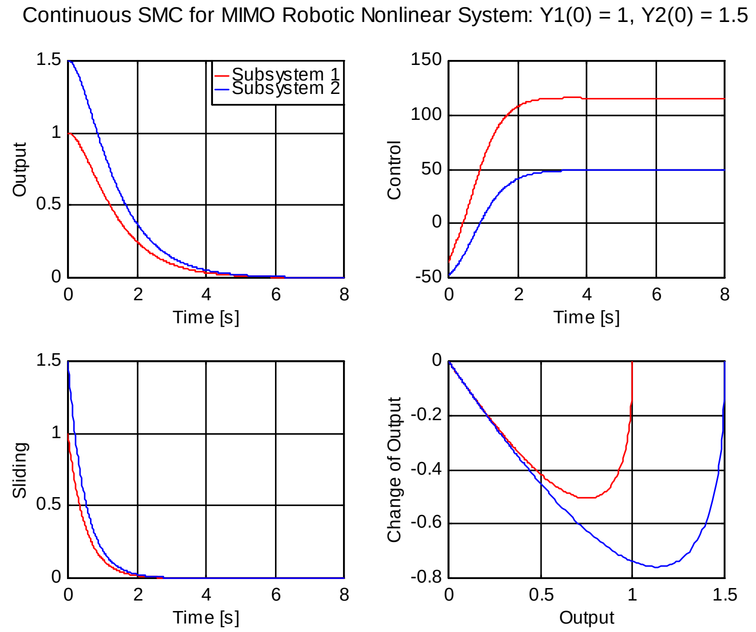


Fig. 7.20: MIMO Continuous Nonlinear SMC for Example 7.5: Y1(0) = 1, Y2(0) = 1.5

7.7.5.4. Performance

- The decoupling is effective as the second output is almost unaffected. The second output is essentially equal to zero if it is initially at zero and under no control. It goes to zero from an initial value if it is under control as required.
- The performance of the continuous SMC is comparable to that of the continuous pseudo-SMC.

7.7.6. Example 7.6: MIMO Uncertain Nonlinear Robot Arms

Consider a robot system in Fu *et al.* 1990

$$\begin{cases} \ddot{y}_1 = \left(M_c + M\right)^{-1}\left[R(M).\dot{y}_2^2 + \left(M_c + M_0 u_1\right)\right], \\ \ddot{y}_2 = I^{-1}(M).\left(-2R(M).\dot{y}_1\dot{y}_2 + I_0 u_2\right), \\ R(M).= M_c y_1 + M\left(y_1 + a\right), \qquad I(M).= J_1 + J_2 + M_c y_1^2 + M\left(y_1 + a\right)^2 \end{cases}$$

Let

$$\begin{cases} x_{11} = y_1 \\ x_{12} = \dot{y}_1 \end{cases}, \qquad \begin{cases} x_{21} = y_2 \\ x_{22} = \dot{y}_2 \end{cases}$$

then with $0\,kg \le M \le 100\,kg$

$$\begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = x_{11}x_{22}^2 + \dfrac{M.x_{22}^2 + 200u_1}{100 + M} \end{cases}, \qquad \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = \dfrac{\begin{Bmatrix} -2\left[100x_{11} + M\left(1 + x_{11}\right)\right]x_{11}x_{12}\cdots \\ +\left[200 + 100x_{11}^2 + 100\left(1 + x_{11}\right)^2\right].u_2 \end{Bmatrix}}{200 + 100x_{11}^2 + M\left(1 + x_{11}\right)^2} \end{cases}$$

- For the subsystem 1:

$$\begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = x_{11}x_{22}^2 + \dfrac{M.x_{22}^2 + 200.u_1}{100 + M} \end{cases}$$

so

$$\dot{s}_1 = h_1\dot{x}_{11} + \dot{x}_{12} = h_1 x_{12} + x_{11}x_{22}^2 + \frac{M.x_{22}^2 + 200.u_1}{100 + M} = \frac{200}{100 + M}\left[u_1 - u_{eq1}\right]$$

thus

$$-u_{eq1} = \frac{100 + M}{200}\left(h_1 x_{12} + x_{11}x_{22}^2\right) + \frac{M}{200}x_{22}^2 = \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle \left(x_{11} + h_1 x_{12}\right) + \left\langle \begin{matrix} 0.5 \\ 0 \end{matrix} \right\rangle x_{22}^2$$

then

$$\begin{cases} \lambda_{H1} = [-2] \Rightarrow \mathbf{H}_1 = [2, \quad 1], & \mathbf{w}_1 = \left[x_{11}, \quad x_{12,} \quad x_{22}^2\right] \\ \mathbf{K}_{e1} = \left[1.5, \quad 1.5 \times \mathbf{H}_1(1) \quad 0.25\right], & \mathbf{K}_{p1} = \left[0.5, \quad 0.5 \times \mathbf{H}_1(1) \quad 0.25\right] \end{cases}$$

- For the subsystem 2:

$$\begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = \dfrac{-2\left[100x_{11} + M(1 + x_{11})\right]x_{11}x_{12} + \left[200 + 100x_{11}^2 + 100(1 + x_{11})^2\right].u_2}{200 + 100x_{11}^2 + M(1 + x_{11})^2} \end{cases}$$

so

$$\dot{s}_2 = h_2\dot{x}_{21} + \dot{x}_{22} = h_2 x_{22} + \frac{-2\left[100x_{11} + M(1 + x_{11})\right]x_{11}x_{12} + \left[200 + 100x_{11}^2 + 100(1 + x_{11})^2\right].u_2}{200 + 100x_{11}^2 + M(1 + x_{11})^2}$$

thus

$$\dot{s}_2 = \frac{200 + 100x_{11}^2 + 100(1 + x_{11})^2}{200 + 100x_{11}^2 + M(1 + x_{11})^2}\left(u_2 - u_{eq2}\right)$$

hence

$$-u_{eq2} = \frac{(200 + M) + 2Mx_{11} + (100 + M)x_{11}^2}{200 + 100x_{11}^2 + 100(1 + x_{11})^2}h_2 x_{22} - \frac{2\left[(100 + M)x_{11} + M\right]x_{11}}{200 + 100x_{11}^2 + 100(1 + x_{11})^2}x_{12}$$

since

$$\frac{200 + 100x_{11}^2 + M(1 + x_{11})^2}{200 + 100x_{11}^2 + 100(1 + x_{11})^2} = \left\langle \begin{matrix} 1 \\ 0.4 \end{matrix} \right\rangle$$

or

$$200 + 100x_{11}^2 + 100(1 + x_{11})^2 > 250 \Rightarrow \frac{1}{200 + 100x_{11}^2 + 100(1 + x_{11})^2} = \left\langle \begin{matrix} 1/250 \\ 0 \end{matrix} \right\rangle$$

so

$$-u_{eq2} = \left\langle \begin{matrix} 0.8h_2 \\ 0 \end{matrix} \right\rangle x_{22} + \left\langle \begin{matrix} 0 \\ -0.8 \end{matrix} \right\rangle x_{11}x_{12} + \left\langle \begin{matrix} 0 \\ -1.6 \end{matrix} \right\rangle x_{12}x_{11}^2 = \left\langle \begin{matrix} 0.4h_2 \\ 0.4h_2 \end{matrix} \right\rangle x_{22} + \left\langle \begin{matrix} -0.4 \\ 0.4 \end{matrix} \right\rangle x_{11}x_{12} + \left\langle \begin{matrix} -0.8 \\ 0.8 \end{matrix} \right\rangle x_{12}x_{11}^2$$

then

$$\begin{cases} \lambda_{H2} = [-2] \Rightarrow \mathbf{H}_2 = [2, \quad 1], & \mathbf{w}_2 = \left[x_{22}, \quad x_{11}x_{12,} \quad x_{12}x_{11}^2\right] \\ \mathbf{K}_{e2} = \left[0.4 \times \mathbf{H}_2(1) \quad 0, \quad 0\right], & \mathbf{K}_{p2} = \left[0.4 \times \mathbf{H}_2(1) \quad 0.4, \quad 0.8\right] \end{cases}$$

7.7.6.1. Robust Discontinuous SMC

Theorem 7.7 yields

$$\begin{cases} u_1 = -\mathbf{K}_{e1}\mathbf{w}_1 - \mathbf{K}_{rp1}.\left|\mathbf{w}_1\right|.\mathrm{sgn}(s_1) \\ \mathbf{K}_{e1} = [1.5, \quad 3, \quad 0.25] \\ \delta_1 = 0.5 \Rightarrow \mathbf{K}_{rp1} = \mathbf{K}_{r1} + \mathbf{K}_{p1} = [\delta_1] + \mathbf{K}_{p1} = [1, \quad 1.5, \quad 0.75] \end{cases}$$

and

$$\begin{cases} u_2 = -\mathbf{K}_{e2}\mathbf{w}_2 - \mathbf{K}_{rp2}.\left|\mathbf{w}_2\right|.\mathrm{sgn}(s_2) \\ \mathbf{K}_{e2} = [0.8, \quad 0, \quad 0] \\ \delta_2 = 1 \Rightarrow \mathbf{K}_{rp2} = \mathbf{K}_{r2} + \mathbf{K}_{p2} = [\delta_2] + \mathbf{K}_{p2} = [1.8, \quad 1.4, \quad 1.8] \end{cases}$$

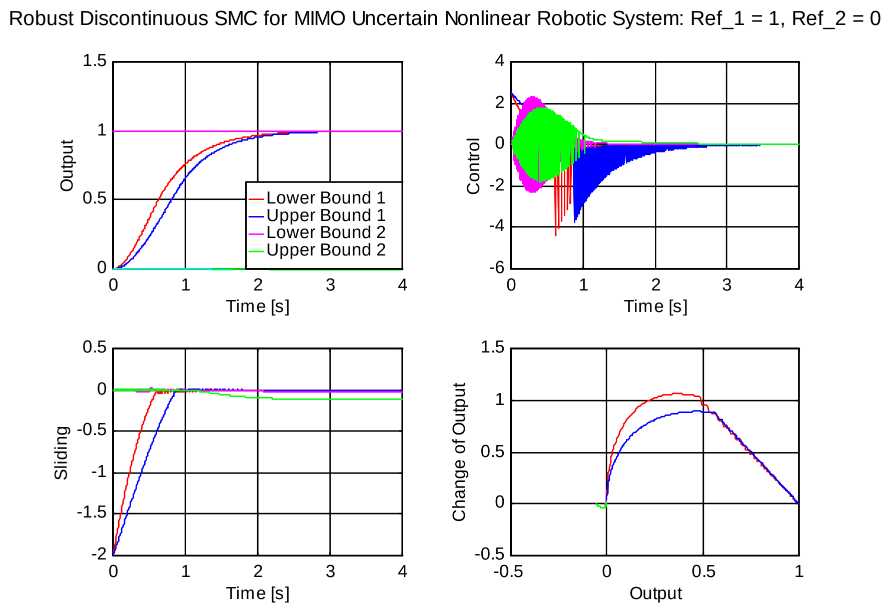


Fig. 7.21: Robust Discontinuous SMC for MIMO Uncertain Nonlinear System in Example 7.6: Ref_1 = 1.0, Ref_2 = 0

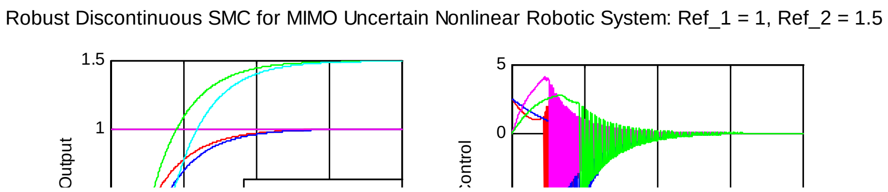


Fig. 7.22: Robust Discontinuous SMC for MIMO Uncertain Nonlinear System in Example 7.6: Ref_1 = 1, Ref_2 = 1.5

7.7.6.2. Robust Continuous Pseudo-SMC

In the above discontinuous functions, let $\operatorname{sgn}(s) \to \tanh(k_s s)$ and choose $k_{s1} = 4,\; k_{s2} = 5$ to have

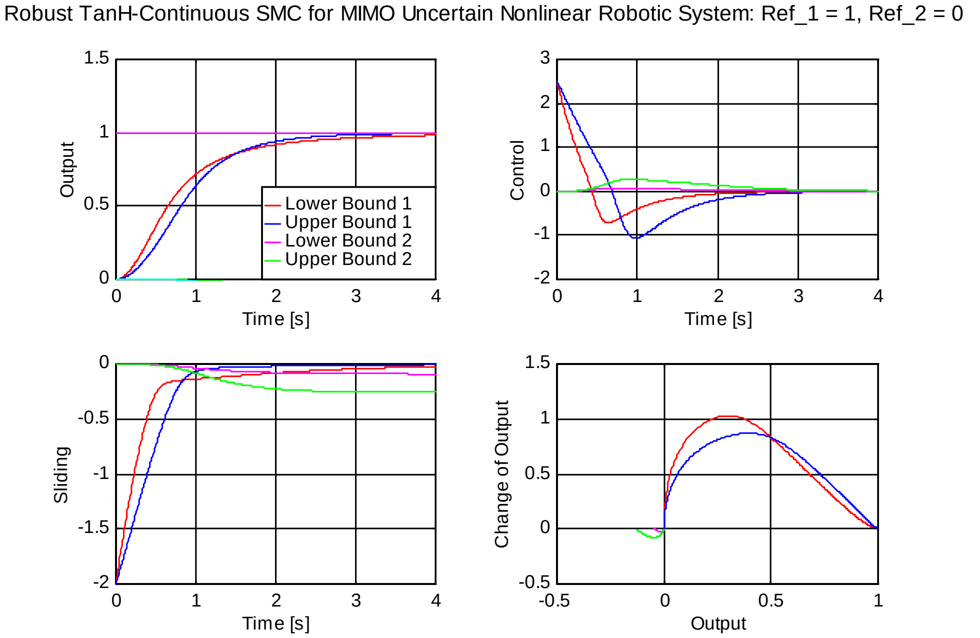


Fig. 7.23: Robust TanH-Continuous SMC for MIMO Uncertain Nonlinear System in Example 7.6: Ref_1 = 1.0, Ref_2 = 0: Steady-State Error

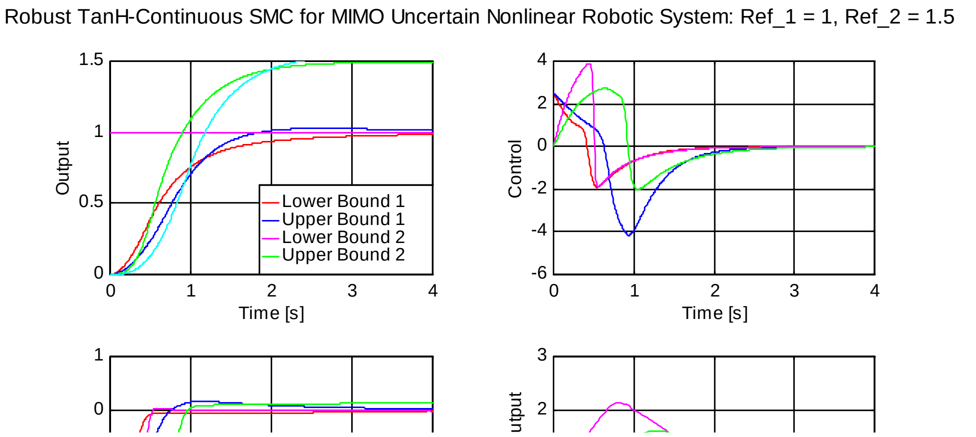


Fig. 7.24: Robust TanH-Continuous SMC for MIMO Uncertain Nonlinear System in Example 7.6; Ref_1 = 1, Ref_2 = 1.5: Steady-State Error

The design scheme in Proposition 3.3 with Err_Tol = 0.05 eliminates the steady-state error as follows

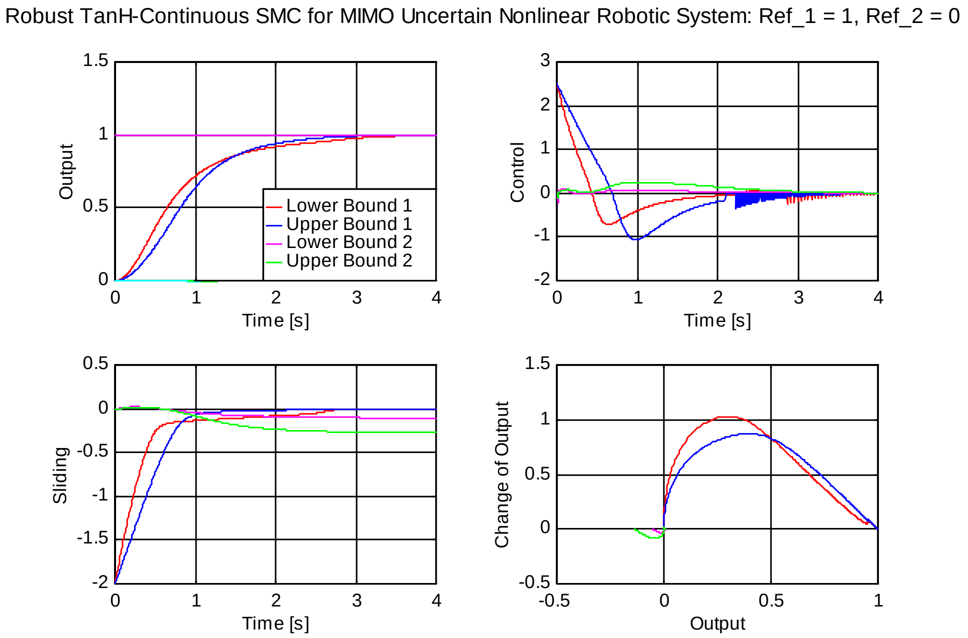


Fig. 7.25: Robust TanH-Continuous SMC for MIMO Uncertain Nonlinear System in Example 7.6; Ref_1 = 1.0, Ref_2 = 0: using Proposition 3.3 with Err_Tol = 0.05 to eliminate Steady-State Error

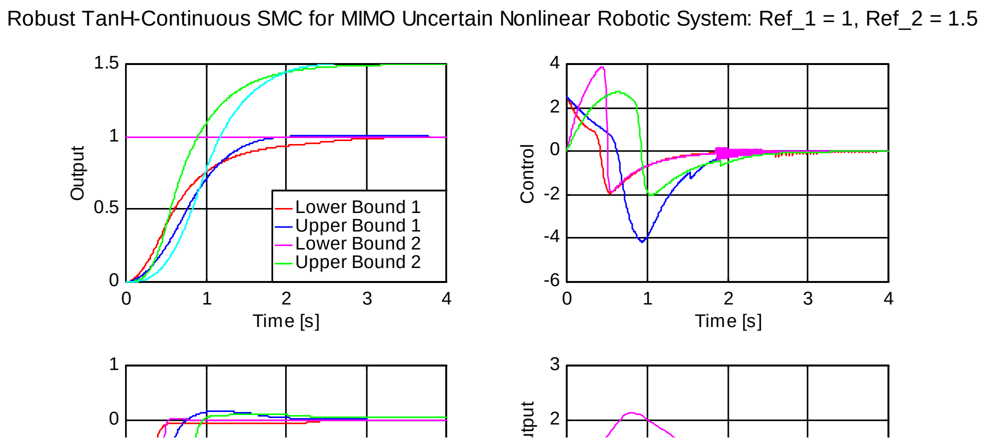


Fig. 7.26: Robust TanH-Continuous SMC for MIMO Uncertain Nonlinear System in Example 7.6; Ref_1 = 1, Ref_2 = 1.5: using Proposition 3.3 with Err_Tol = 0.05 to eliminate Steady-State Error

Remark 7.9: Steady-State Error due to Low Sliding Gain and Efficiency of Design Scheme (Proposition 3.3)

- As expected in Section 3.5.5, low sliding gains cause steady-state errors using a sliding function to replace the switching function in eliminating the chattering;
- The design scheme in Proposition 3.3 is efficient in eliminating these steady-state errors

7.7.6.3. Performance

The decoupling is effective as the second output is almost unaffected. The second output is essentially equal to zero if it is initially at zero and under no control. It goes to desired position if it is under control as required.

7.8. CONCLUSION

For a nonlinear system without perturbation, we can get a continuous SMC. But for a uncertain dynamical nonlinear system, we can get a continuous pseudo-SMC only. Note that the continuous pseudo-SMC is also applicable to a nonlinear system without any perturbation, so we have got continuous pseudo-SMC controllers for all numerical examples above to compare with available continuous SMC controllers, and we have found out that the difference is negligible.

For the general case, if the system Jacobian is non-singular, we can use the decoupling technique to deal with a MIMO in the unified manner as with a SISO. Otherwise, we have to use the complicated *Structure-Algorithm* (Silverman 1969) and the error dynamic equation (Chen *et al.* 1992). We have dealt with a system in Chen *et al.* 1992 by using the simple decoupling technique where any perturbation can be taken into account. Note that this system has been considered by using the structure-algorithm in the above work where it is not clear how to take perturbations into account.

All the numerical examples above have the relative degree equal to the system order, so they are in the canonical form, thus the matching condition is satisfied, so the system stability is guaranteed by the stable hyperplane. If the relative degree is less than the system order, a stability test can be made for a linear system, but not for a nonlinear system. If this is the case, that nonlinear system should be linearized and a stability test can be done where the nonlinearity is considered as uncertainties.

A SMC design involves 2 design stages: hyperplane design and controller design. The hyperplane design is independent of the controller design because it is designed regardless the implementation of the controller in terms of its form (*continuous SMC* or *discontinuous SMC* or *continuous pseudo-SMC*). Based on the *nominal model,* a hyperplane is designed in such a way that it is stable, thus it is chosen to be linear with its Hurwitz eigenvalues, hence a *stable hyperplane* even for a nonlinear system. Based on the *actual model* where any potential perturbation is taken into account, a control function is designed in such a way that it satisfies the sliding condition to make the system states slide on the hyperplane even under perturbations. Only then the system is stable if the sliding mode is stable.

For a linear or nonlinear *canonical* system, the hyperplane is independent of the system parameters, so a stable hyperplane does guarantee the system stability. For a *linear* system, if the matching condition is satisfied, then a stable hyperplane also guarantees the system stability; otherwise it is necessary to check if the sliding-eigenvalues are Hurwitz. For a *nonlinear* system, it should be linearized in order to calculate the sliding-eigenvalues where the nonlinearity is considered as uncertainties, then it is necessary to check if the sliding-eigenvalues of the linearized system are Hurwitz in the operating range. Note that the *invariance conditions* in Section 4.2 are valid for linear systems, not for nonlinear systems, because they are based on linear systems. It is crucial to note that the *invariance property* is valid for both canonical linear and nonlinear systems, and for non-canonical linear systems satisfying the matching condition. This property means that the sliding dynamic is invariant to perturbations. For a linear system, the sliding-eigenvalues are of the actual model, while the hyperplane-eigenvalues are the desired sliding-eigenvalues of the nominal model. In other words, every actual model has its own sliding plane and if the invariance condition is satisfied then all these sliding planes coincide with the hyperplane, hence the invariance property. For canonical linear or nonlinear systems, there is only one hyperplane possible, so the invariance property must be implied.

Chapter 7

General Sliding-Mode Controller Design for MIMO Uncertain Nonlinear Systems

Chapter 8

Advanced Sliding-Mode Controller Design: Experimental Results

8.1. Introduction

In this chapter, results from experiments of Ball-Hoop system are presented to validate our anticipations in theory such as infinite gain of the switching function may cause excitation of unmodelled high-frequency; performances of the saturate, unitvector and hyperbolic tangent functions; performances of continuous SMC, continuous pseudo-SMC (Tanh SMC) and sliding-mode fuzzy control.

One of the consequences of the trend towards the high speed transportation of bulk liquids is that the influence of the cargo upon the vehicle must be fully accounted for. This has always been true in the static sense, but in recent years the dynamical interaction between the material being transported and its container has grown in importance. In fact, the behaviour of a vehicle during a manoeuvre is very often a joint function of the liquid cargo and vehicle dynamics. This problem, referred to as "*liquid slop*", is known to be a problem in:

(i) High speed road and rail transport of bulk liquids
(ii) Maritime transportation, especially in oil tankers
(iii) Liquid fuelled missiles

Actually, "*liquid slop*" is not confined to vehicular transport, any arrangement which involves the rapid movement of large quantities of fluid can exhibit the characteristic oscillations which are associated with "*liquid slop*". The pumping of concrete in the civil engineering and building industry is a fair example.

An essential aim of the Ball-Hoop system is to illustrate the dynamical behaviour and control problems associated with liquid slop. In particular, if a cylindrical vessel is considered, then to a first approximation, the essential dynamical character of liquid movement in the cylinder (Fig. 8.1.a) is captured by the motion of a body rolling inside a hoop (Fig. 8.1.b).

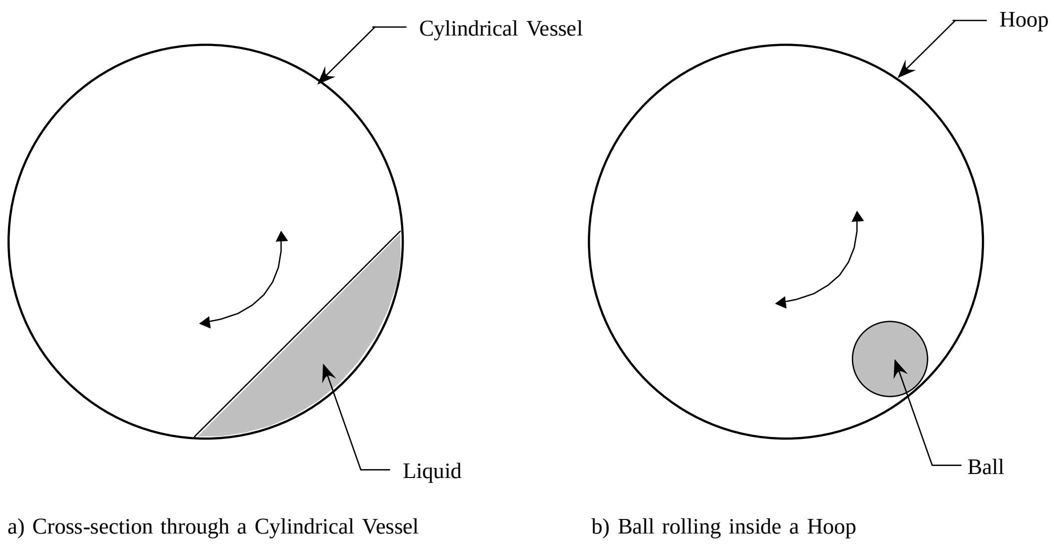


Fig. 8.1: Ball-Hoop motion

8.2. SYSTEM DESCRIPTION

In the system the vehicle motion is introduced by allowing the hoop to rotate under the action of a direct drive servo-motor, while the liquid motion is modelled by the oscillations of a ball rolling in the inner periphery of the hoop.

The Ball-Hoop system consists of a D.C. servo-motor which is equipped with an integral tachometer and a hoop angular position transducer mounted in a vertical frame. A large annular hoop is directly fixed to the motor shaft and constitutes an inertial load on the motor. A metal ball is placed inside the hoop so that it rolls in the groove on the inner periphery of the hoop. The ball angular position is measured using a potentiometer as the ball position transducer mounted on the front of, and coaxial with, the hoop. A wiper assembly is connected to the potentiometer shaft and registers on the ball via a yoke. This provides a measurement of the angular deviation of the ball from vertical, which is referred to as the "*slop angle*".

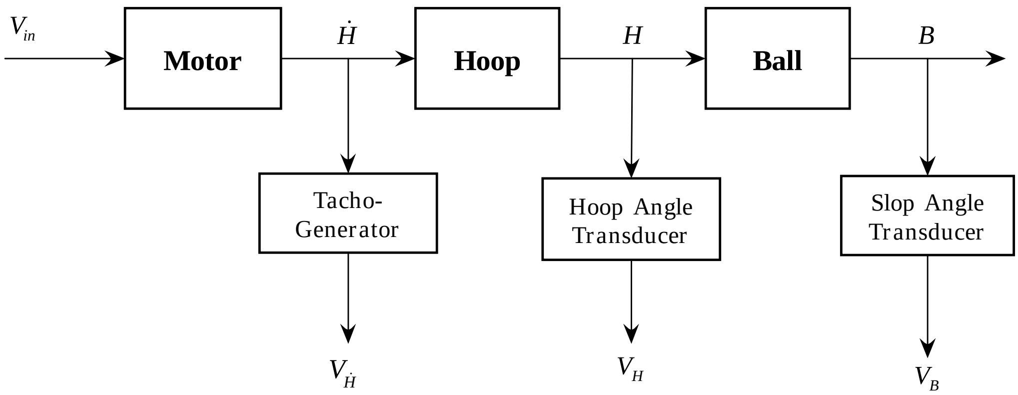


Fig. 8.2: Block diagram of the Ball-Hoop system

where

B : Angular deviation of the ball from the vertical (slop angle)

H : Angular position of the hoop

$\dot{H}$: Angular velocity of the hoop

8.3. INSTRUMENTATION

The *instrumentation box* has

1 input : Servo-motor drive voltage

3 outputs: (i) Hoop velocity signal, derived from tachometer

(ii) Hoop angular position signal, derived from the rotary position transducer attached to the rear motor shaft

(iii) Ball angular position signal, referred to as the "slop angle", derived from the rotary potentiometer mounted in front of the hoop.

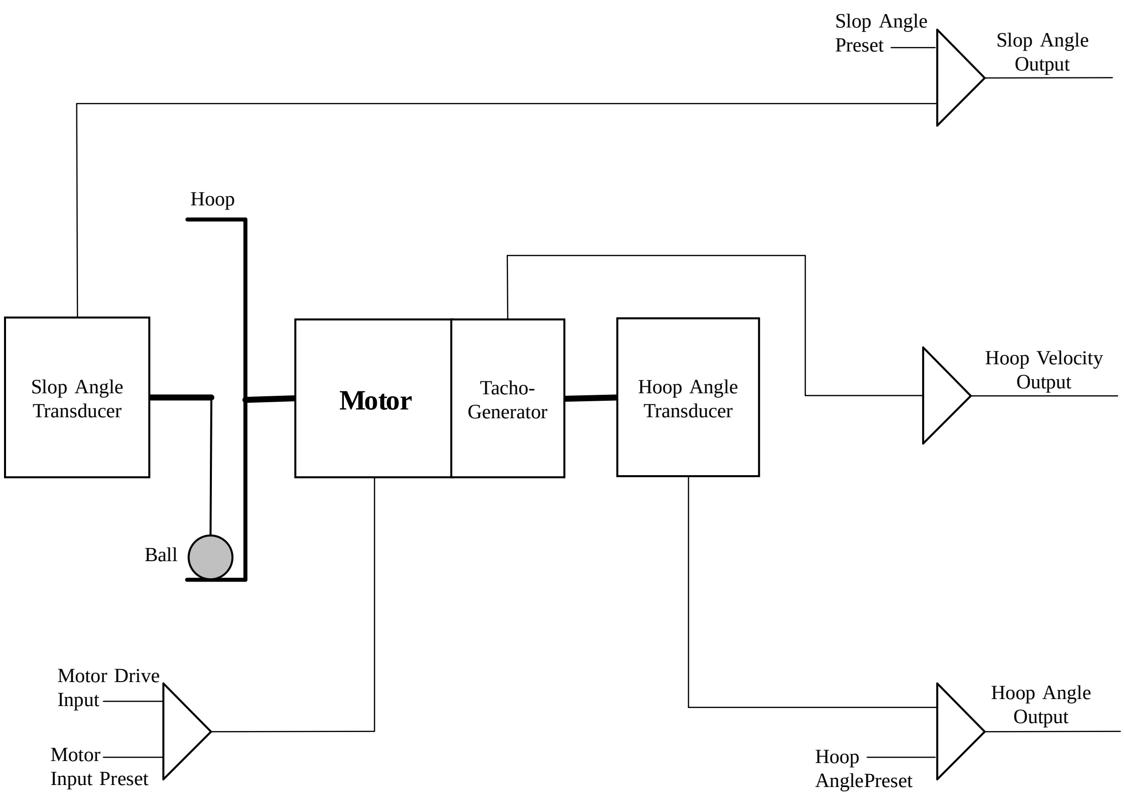


Fig. 8.3: Instrumentation box

The instrumentation associated with the Ball-Hoop system is typical of that found in D.C. servo position control system. Namely,

(i) A D.C. servo-motor

(ii) Tacho-generator

(iii) Angular position transducer

In addition, the system is equipped with a rotary potentiometer for measuring the angular deflection of the ball.

8.4. SYSTEM IDENTIFICATION

A theoretical and practical system identifications will be presented in this section. In the theoretical method, based on only one typical actual response from some experiments, a model verification will be used to identify all unknown model parameters to obtain a system model. The practical method has been presented in Chapter 6 where the fuzzy inference has been used to obtain a system model based on numerous actual responses from some experiments, this is a fuzzy modelling.

8.4.1. Theoretical System Identification

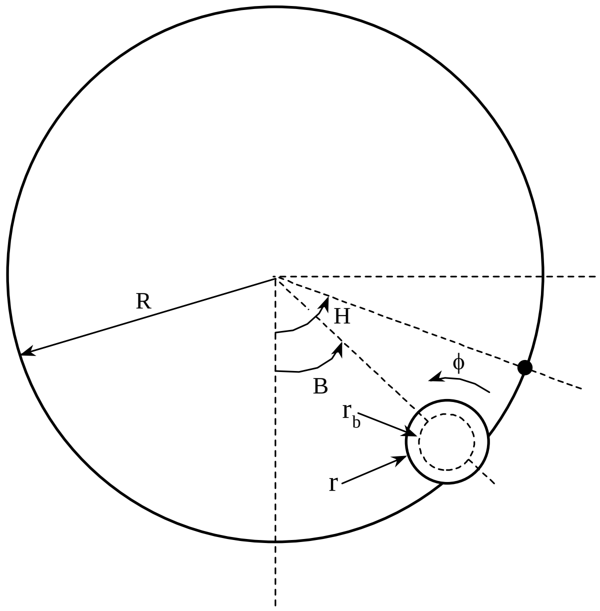


Fig. 8.4: Description of Ball-Hoop Dynamics

where

I_a: Moment of Inertia of the Hoop

I_b: Moment of Inertia of the Ball

M_b: Mass of the Ball

B_b: Coefficient of Rolling Friction of the Ball

B_m: Coefficient of Rotational Friction of the Motor Assembly

T_h: Torque

R: Radius of the Hoop

r: Radius of the Ball

B: Ball angle against the vertical axis

H: Hoop angle against vertical axis

We will use Lagrange approach to derive theoretically a mathematical model of the system using the following extended Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial J}{\partial \dot{q}_i} = F_i, \quad L = T - U \tag{8.1}$$

where

T : total co-kinetic energy

U : total potential energy

J : total co-content energy

q_i : generalized coordinate

Equation for the transformation of coordinates:

$$\phi = \frac{R}{r}(H - B) \tag{8.2}$$

Translational velocity of the Ball

$$v = (R - r)\dot{B} \tag{8.3}$$

The hoop rotates only, but the ball rotates and translates as well. The total *co-kinetic energy* of the system

$$T = \frac{1}{2} I_a \dot{H}^2 + \frac{1}{2} I_b \dot{\phi}^2 + \frac{1}{2} M_b v^2 \tag{8.4}$$

Eqs.(8.2) and (8.3) give

$$T = \frac{1}{2}\left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - I_b\left(\frac{R}{r}\right)^2 \dot{H}\dot{B} + \frac{1}{2}\left[I_b\left(\frac{R}{r}\right)^2 + M_b(R - r)^2\right]\dot{B}^2 \tag{8.5}$$

The total *potential energy* of the system:

$$U = M_b g.(R - r).(1 - \cos B) \tag{8.6}$$

For the ball and hoop, although the ball translates, there is no translational friction for the ball. The total *co-content energy* of the system

$$J = \frac{1}{2} B_m \dot{H}^2 + \frac{1}{2} B_b \dot{\phi}^2 \tag{8.7}$$

Eq.(8.2) yields

$$J = \frac{1}{2}\left[B_m + B_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - B_b\left(\frac{R}{r}\right)^2 \dot{H}\dot{B} + \frac{1}{2} B_b\left(\frac{R}{r}\right)^2 \dot{B}^2 \tag{8.8}$$

Hence the system Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) + \frac{\partial J}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = F_n \tag{8.9}$$

where

$$L = T - U = \frac{1}{2}\left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - I_b\left(\frac{R}{r}\right)^2 \dot{H}\dot{B} + \frac{1}{2}\left[I_b\left(\frac{R}{r}\right)^2 + M_b(R - r)^2\right]\dot{B}^2 - M_b g.(R - r).(1 - \cos B)$$

(8.10)

then

$$\frac{\partial L}{\partial \dot{H}} = \left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\dot{H} - I_b\left(\frac{R}{r}\right)^2 \dot{B} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{H}}\right) = \left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\ddot{H} - I_b\left(\frac{R}{r}\right)^2 \ddot{B}$$

and

$$\frac{\partial L}{\partial \dot{B}} = -I_b\left(\frac{R}{r}\right)^2 \dot{H} + \left[I_b\left(\frac{R}{r}\right)^2 + M_b(R-r)^2\right]\dot{B} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{B}}\right) = -I_b\left(\frac{R}{r}\right)^2 \ddot{H} + \left[I_b\left(\frac{R}{r}\right)^2 + M_b(R-r)^2\right]\ddot{B}$$

and

$$\frac{\partial L}{\partial H} = 0$$

$$\frac{\partial L}{\partial B} = -M_b g.(R-r)\sin B$$

Eq.(8.8) gives

$$\frac{\partial J}{\partial \dot{H}} = \left[B_m + B_b\left(\frac{R}{r}\right)^2\right]\dot{H} - B_b\left(\frac{R}{r}\right)^2 \dot{B}$$

$$\frac{\partial J}{\partial \dot{B}} = -B_b\left(\frac{R}{r}\right)^2 \dot{H} + B_b\left(\frac{R}{r}\right)^2 \dot{B}$$

Thus

(1) $q_1 = H$

$$\left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\ddot{H} + \left[B_m + B_b\left(\frac{R}{r}\right)^2\right]\dot{H} - \left(\frac{R}{r}\right)^2\left(I_b\ddot{B} + B_b\dot{B}\right) = T \quad (8.11)$$

(2) $q_2 = B$

$$\left(\frac{R}{r}\right)^2\left(I_b\ddot{H} + B_b\dot{H}\right) - \left[I_b\left(\frac{R}{r}\right)^2 + M_b(R-r)^2\right]\ddot{B} - B_b\left(\frac{R}{r}\right)^2\dot{B} - M_b g(R-r)\sin B = 0 \quad (8.12)$$

8.4.1.1. Ball-Hoop Transfer Function

Assume that

$$\begin{cases} \sin B \approx B \\ r \approx r_b \\ I_b \approx \dfrac{2}{5}M_b r_b^2 : \text{ball is a solid sphere} \end{cases}$$

then from Eq.(8.12)

$$\left(\ddot{H} + \frac{B_b}{I_b}\dot{H}\right) - \frac{7}{2}\ddot{B} - \frac{B_b}{I_b}\dot{B} - \frac{5}{2R}gB = 0$$

or

$$\frac{B(s)}{H(s)} = \frac{2}{7}\frac{s\left[s + \frac{5}{2}\left(\frac{B_b}{M_b r^2}\right)\right]}{s^2 + \frac{5}{7}\left(\frac{B_b}{M_b r^2}\right)s + \frac{5g}{7r}} \quad (8.13)$$

where $g = 9.81$ m/s^2

By a simple measurement of Hoop radius: R = 8.75 cm

In fact, r ≈ 0.70 cm, so R >> r as the above assumption

r_b ≈ 0.75 cm, so r_b ≈ r as the above assumption

From Eq.(8.13):

$$\frac{B(s)}{H(s)} = \frac{2}{7}\frac{s\left(s+\frac{A}{5}\right)}{s^2+\frac{A}{7}s+80} \tag{8.14}$$

where :

$$A = \frac{5B_b}{M_b r^2} \tag{8.15}$$

Hence

$$\omega_n = \sqrt{80} = 8.944 rad/s$$

Let

$$A/7 = 2\delta\omega_n = 17.888\,\delta \approx 18\,\delta \tag{8.16}$$

where : δ is Damping Factor

So

$$A/5 = (7/5)(A/7) \approx 25\,\delta \tag{8.17}$$

and Eqs.(8.14) & (8.15) give

$$\delta = \frac{5B_b}{126 M_b r^2} = f(B_b) \tag{8.18}$$

Eq.(8.13) yields

$$\frac{B(s)}{H(s)} = 0.286\frac{s(s+25\delta)}{s^2+18\delta s+80} \tag{8.19}$$

Therefore, with only 1 simple measurement, we have got the transfer function with only 1 unknown δ : Damping Factor.

8.4.1.2. DC Servo Motor Transfer Function

A servo motor has a transfer function as

$$\frac{H(s)}{V_{in}(s)} = \frac{K_m}{s\left(s+\frac{1}{\tau_m}\right)} \tag{8.20}$$

8.4.1.3. Model Verification

We first consider each motor and ball subsystems separately, then combine them to see the effect of interactions.

(a) Separated System

Using a step of 2, we obtain the following model for the motor

$$G_m(s) = \frac{1.05}{s+1.5} \times \frac{85}{s} \tag{8.21}$$

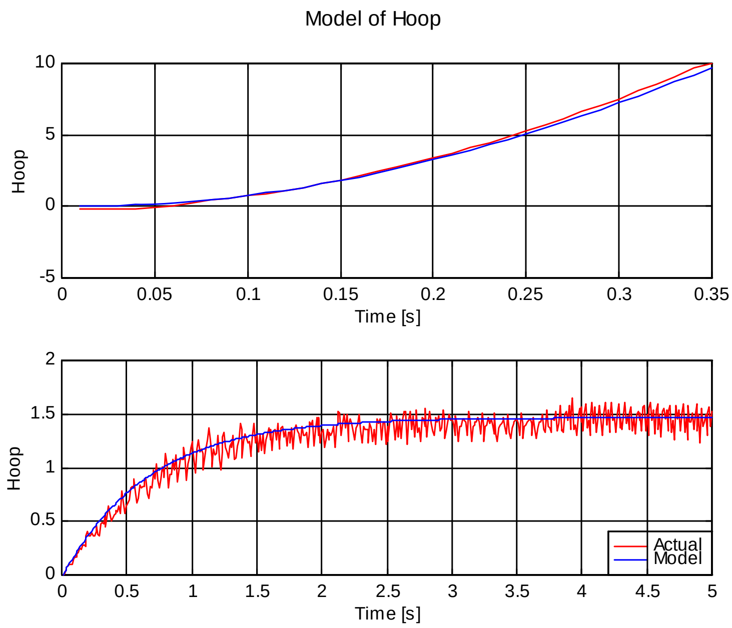


Fig. 8.5: Modelling Motor Subsystem

and for ball-hoop subsystem

$$G_b^*(s) = \frac{0.3(s+1)}{s^2 + 0.72s + 81} \tag{8.22}$$

it is very close to Eq.(8.18) where $\delta = 0.04$.

(b) Combined System

The ball-hoop subsystem is driven by the motor, due to interaction between them, the ball-hoop subsystem is modified to be

$$G_b(s) = \frac{-2.5s^2}{s^2 + 0.84s + 110} \tag{8.23}$$

Note the different numerators in Eqs. (8.22) & (8.23).

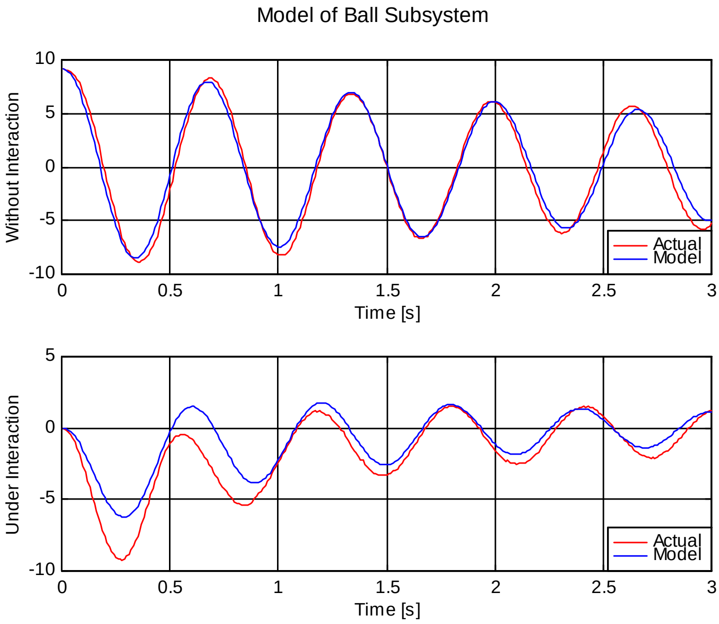


Fig. 8.6: Modelling Individual Ball-Hoop Subsystem

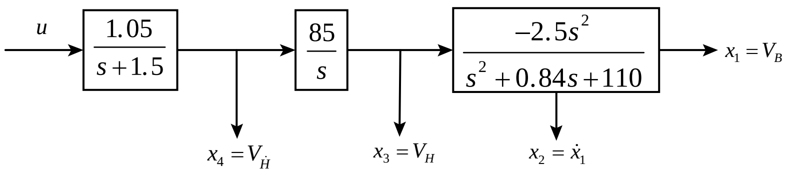


Fig. 8.7: Ball-Hoop Model by a Typical Step Response

8.4.2. Practical System Identification

From Chapter 6, a fuzzy model is found as

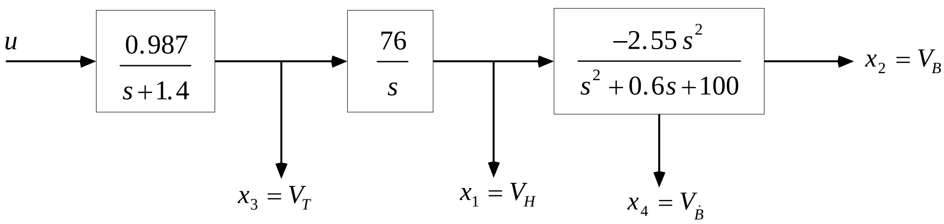


Fig. 8.8: Fuzzy Model of Ball-Hoop System

8.5. DIGITAL SLIDING-MODE CONTROLLER-OBSERVER DESIGNS

All sliding-mode controllers (SMC) will be designed and implemented: switching discontinuous SMC, saturate and tanh pseudo-continuous SMC, linear continuous SMC, and sliding-mode fuzzy controller. The Ball-Hoop system is an 4-th order system, however there are only 3 states available: hoop position , hoop velocity (tacho) and ball position. An observer is thus required to estimate the missing ball velocity and this observer will be used for all types of SMC. A PC and an ADC card will be used to implement controller-observer, so discrete-time controller-observer will be designed.

8.5.1. Robust Sliding-Mode Observer Design

For the model whose system states defined in Fig.8.8, a state-space model is

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \Delta\tilde{\mathbf{A}}\right)\mathbf{x} + \left(\mathbf{B} + \Delta\tilde{\mathbf{B}}\right)u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 77.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1.375 & 0 \\ 0 & -121 & 265.625 & -0.66 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 0 \\ 0 & 11 & -15.625 & 0.03 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.0050 \\ -192.2500 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -0.395 \\ 70.25 \end{bmatrix}$$

choose $\lambda_H = [-8, \quad -8, \quad -8]$ then

$$\mathbf{H} = [0.0692 \quad -0.1176 \quad 2.0204 \quad 0.0040]$$

thus

$$\mathbf{K}_e = [0 \quad -0.4642 \quad 3.8329 \quad -0.1202]$$

choose $T_s = 0.005\ s$, since it is 25 faster than the desired dynamics (λ_H), then

$$\mathbf{A}_d = \begin{bmatrix} 1 & 0 & 0.3862 & 0 \\ 0 & 0.9985 & 0.0033 & 0.0050 \\ 0 & 0 & 0.9931 & 0 \\ 0 & -0.6037 & 1.3207 & 0.9952 \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} 0.0010 \\ -0.0024 \\ 0.0050 \\ -0.9557 \end{bmatrix}$$

and

$$\Delta\mathbf{A}_d = \begin{bmatrix} 0 & 0 & 0.0126 & 0 \\ 0 & 0.0001 & -0.0002 & 0 \\ 0 & 0 & 0.0006 & 0 \\ 0 & 0.0548 & -0.0771 & 0.0003 \end{bmatrix}, \quad \Delta\mathbf{B}_d = \begin{bmatrix} -0.0004 \\ 0.0009 \\ -0.0020 \\ 0.3489 \end{bmatrix}$$

and choose the dynamics of a robust *discrete-time* sliding-mode observer 3 times faster than those of sliding-mode controller, thus $\lambda_{HO} = [-24 \quad -24 \quad -24] \;\Rightarrow\; \lambda^*_{HO} = [0.8869 \quad 0.8869 \quad 0.8869]$, then

$$\mathbf{L} = \begin{bmatrix} 0.3989 & 0 \\ 0.0165 & 1.3607 \\ 0.3282 & 0 \\ 1.2531 & 4.7037 \end{bmatrix} \;\Rightarrow\; \tilde{\mathbf{A}} = \bar{\mathbf{A}} - \mathbf{LC} = \begin{bmatrix} 0.6011 & 0 & 0.3862 & 0 \\ -0.0165 & -0.3622 & 0.0033 & 0.0050 \\ -0.3282 & 0 & 0.9931 & 0 \\ -1.2531 & -5.3074 & 1.3207 & 0.9952 \end{bmatrix}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the observer equation is

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}.\tilde{\mathbf{x}}(k) + \mathbf{B}_d.u(k) + \mathbf{L}.y(k)$$

This observer equation will be used to estimate the ball velocity for all the following SMC's.

8.5.2. Robust Sliding-Mode Controller Designs

All controller types will use the same hyperplane above.

8.5.2.1. Robust Switching SMC

We have

$$\mathbf{K}_p = \begin{bmatrix} 0 & 0.0596 & 0.6580 & 0.0002 \end{bmatrix}$$

choose $\delta = 6$, then

$$\mathbf{K}_r = \begin{bmatrix} 0.4151 & 0.7057 & 12.1223 & 0.0241 \end{bmatrix}$$

then the simulation results are

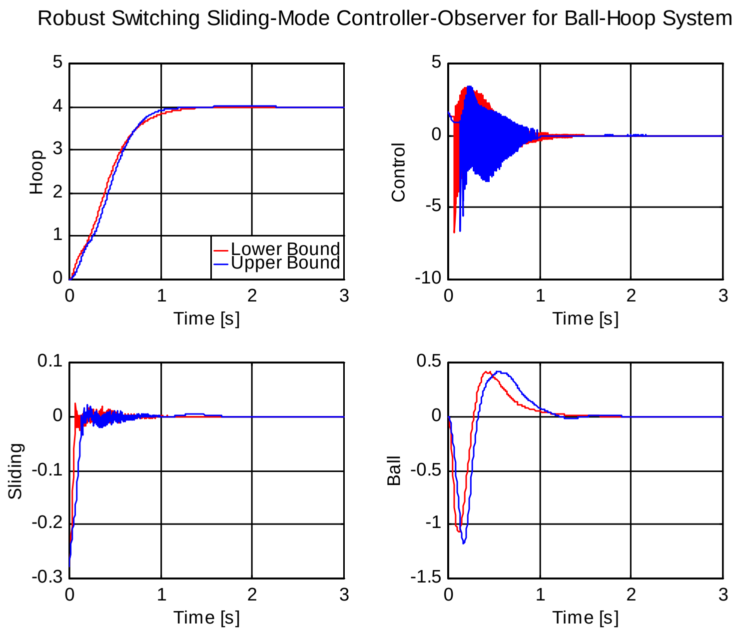
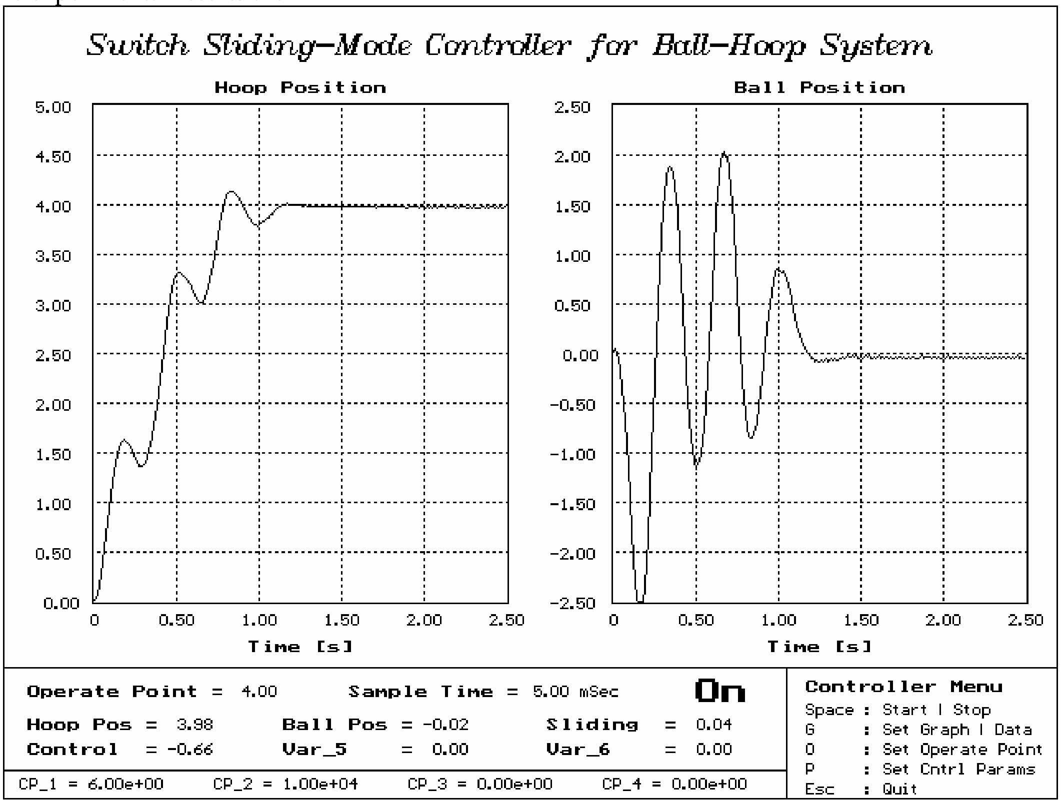


Fig. 8.9: Robust Switching Sliding-Mode Controller-Observer Design for Ball-Hoop System

and the experimental results are



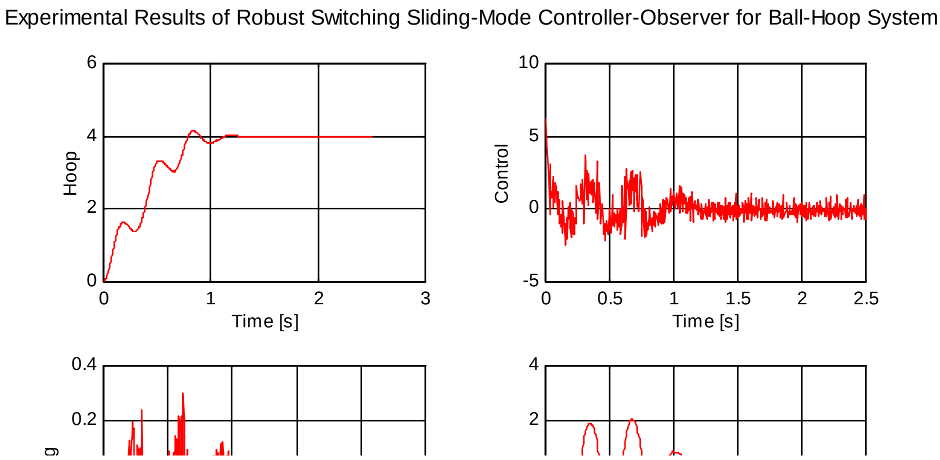


Fig. 8.10: Experimental Results of Robust Switching Sliding-Mode Controller-Observer for Ball-Hoop System

Remark 8.1: Excitation of Unmodelled High-Frequency due to Switching Function

Due to infinite gain of switching function, the unmodelled high-frequency dynamics are excited to cause oscillations.

8.5.2.2. Robust Saturate SMC

Controller parameters are the same as the switching SMC above where

$$\mathrm{sign}(s) \to \mathrm{sat}(s, k_s), \quad k_s = 30$$

then the simulation results are

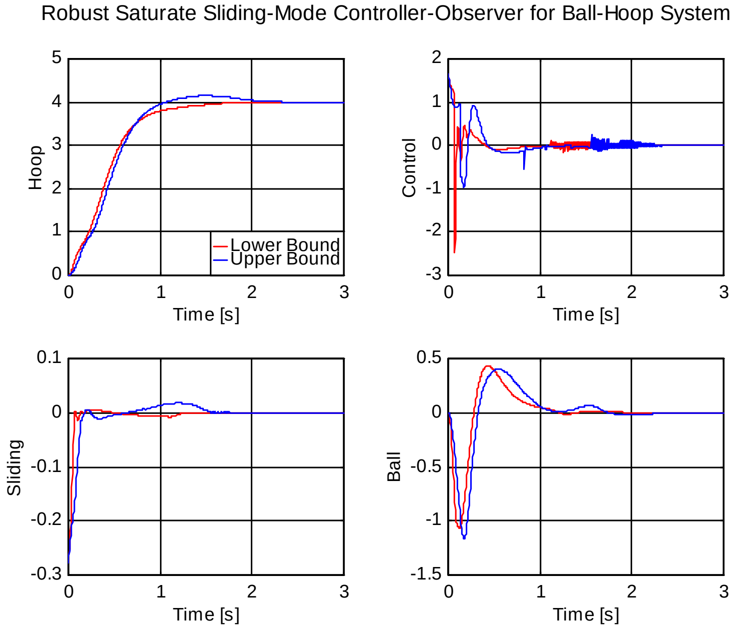
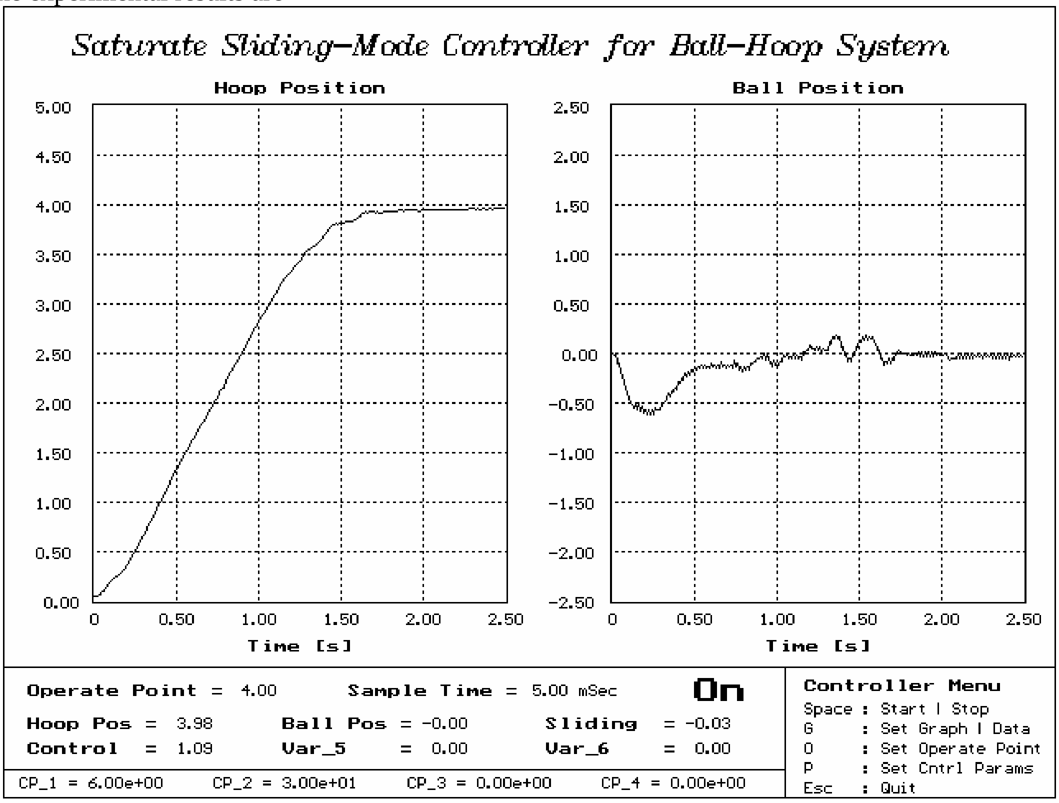


Fig. 8.11: Robust Saturate Sliding-Mode Controller-Observer Design for Ball-Hoop System

and the experimental results are



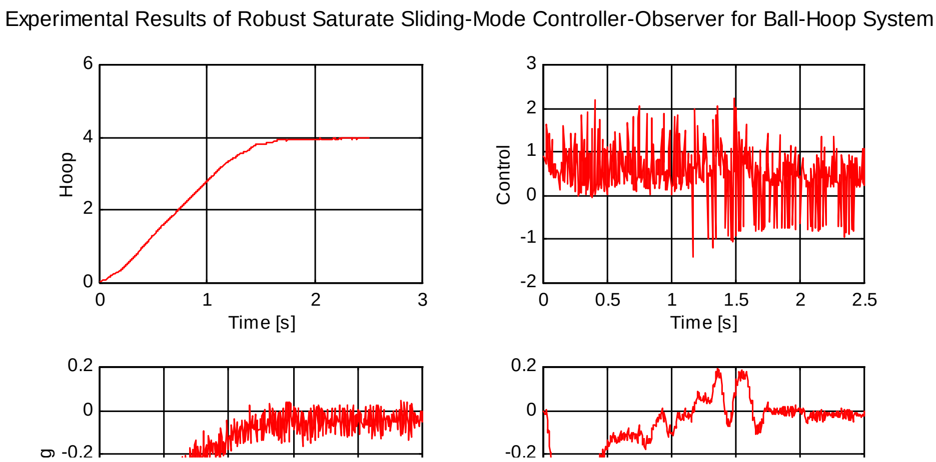


Fig. 8.12: Experimental Results of Robust Saturate Sliding-Mode Controller-Observer for Ball-Hoop System

Remark 8.2: Slow-Down System Response due to Low Sliding Gain

In experiments, the sliding gain has to reduce to 0.3 for alleviating the chattering and the system is slow down due to this lower gain as expected in theory (Section 3.5.5)

8.5.2.3. Robust Unitvector SMC

Controller parameters are the same as the switching SMC above where

$$\mathrm{sign}(s) \rightarrow \mathrm{smt}(k_s, s), \quad k_s = 120$$

then the simulation results are

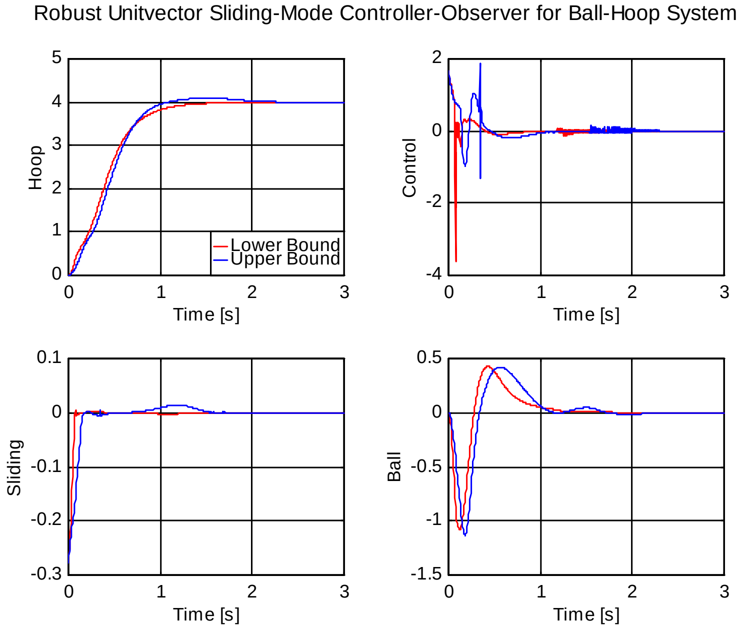
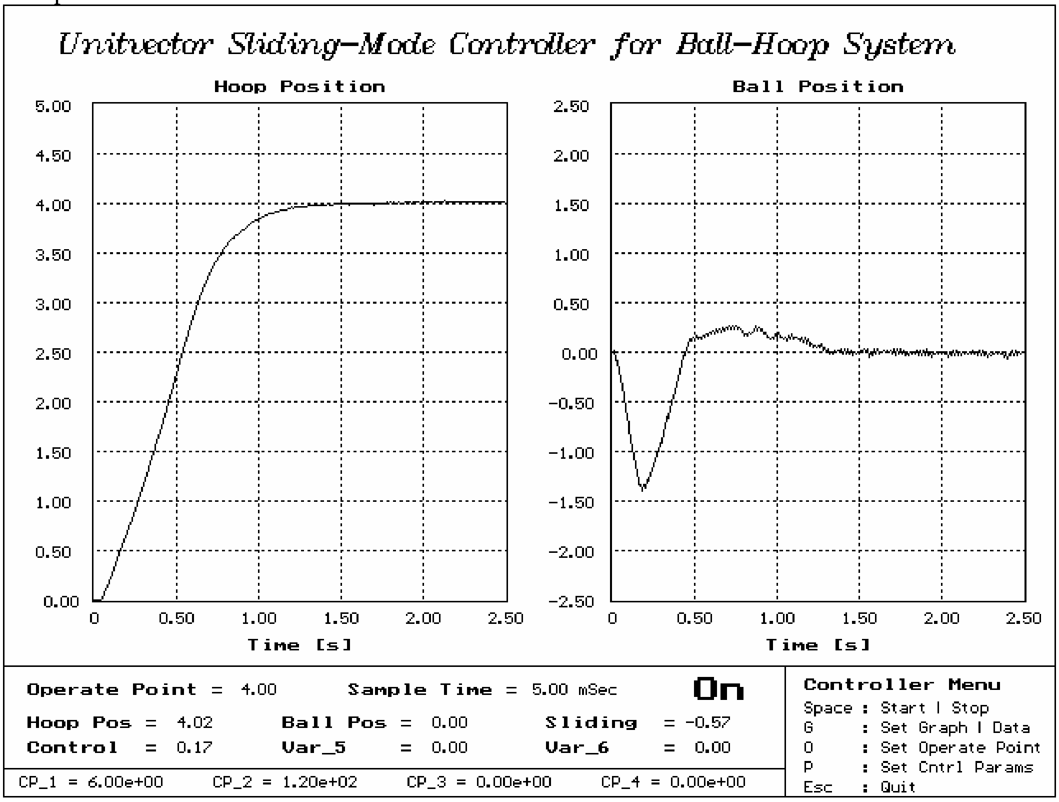


Fig. 8.13: Robust Unitvector Sliding-Mode Controller-Observer Design for Ball-Hoop System

and the experimental results are



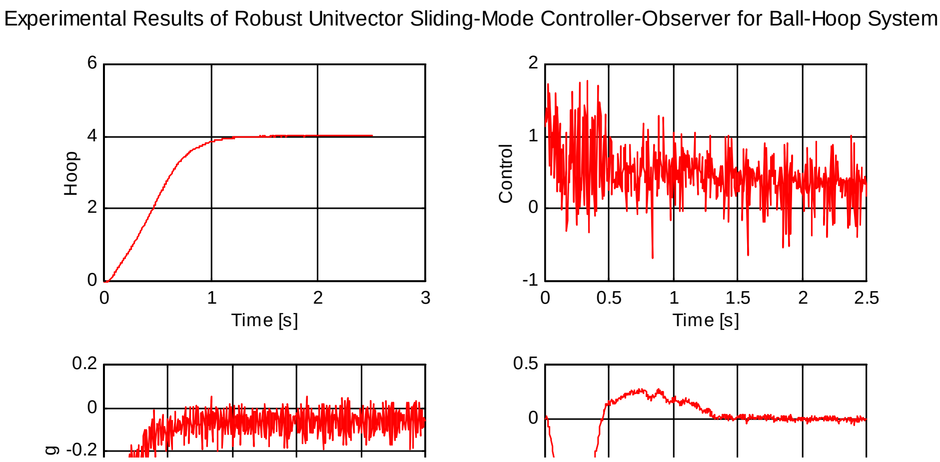


Fig. 8.14: Experimental Results of Robust Unitvector Sliding-Mode Controller-Observer for Ball-Hoop System

8.5.2.4. Robust TanH SMC

Controller parameters are the same as the switching SMC above where

$$\mathrm{sign}(s) \to \tanh(k_s s), \quad k_s = 50$$

then the simulation results are

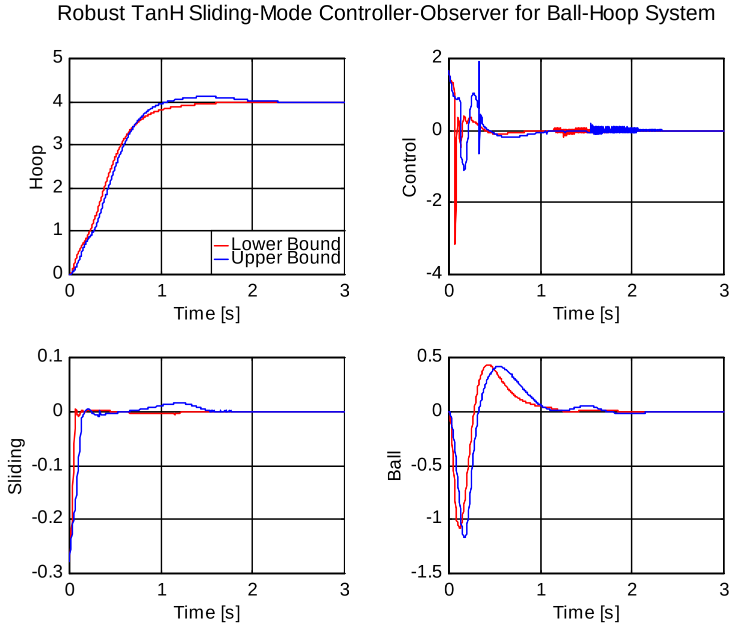
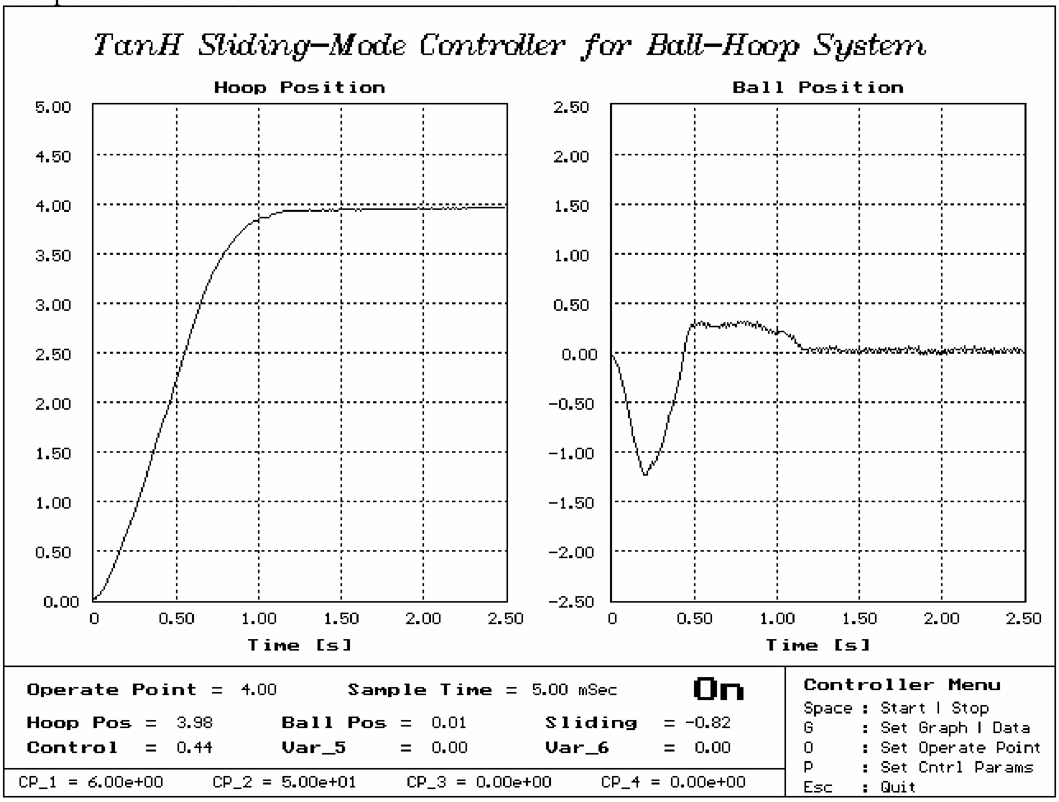


Fig. 8.15: Robust TanH Sliding-Mode Controller-Observer Design for Ball-Hoop System

and the experimental results are



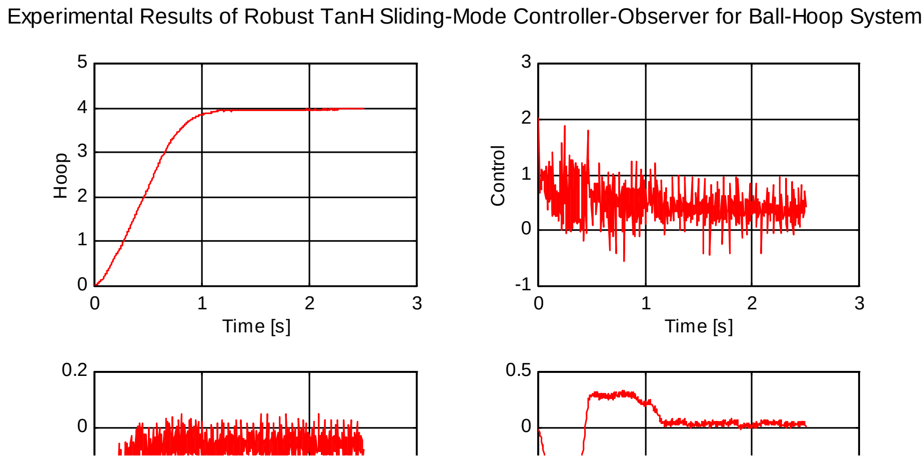


Fig. 8.16: Experimental Results of Robust TanH Sliding-Mode Controller-Observer for Ball-Hoop System

Remark 8.3: Performance of Saturate and TanH SMC

As mentioned in Section 3.5.5, the sliding gain of a TanH SMC can be larger than that of a Saturate SMC. In this experiment, this gain has been chosen 50 for TanH SMC and 30 for Saturate SMC, thus the system response of TanH SMC is faster than Saturate SMC.

8.5.2.5. Robust Linear SMC

We have

$$\mathbf{K}_p = \begin{bmatrix} 1.4101 & -3.1323 & 47.4146 & 0.0932 \end{bmatrix}$$

choose $\delta = 14$, then

$$\mathbf{K}_r = \begin{bmatrix} 0.9686 & -1.6466 & 28.2853 & 0.0563 \end{bmatrix}$$

thus

$$\mathbf{K} = \begin{bmatrix} 2.3787 & -5.2431 & 79.5329 & 0.0293 \end{bmatrix}$$

then the simulation results are

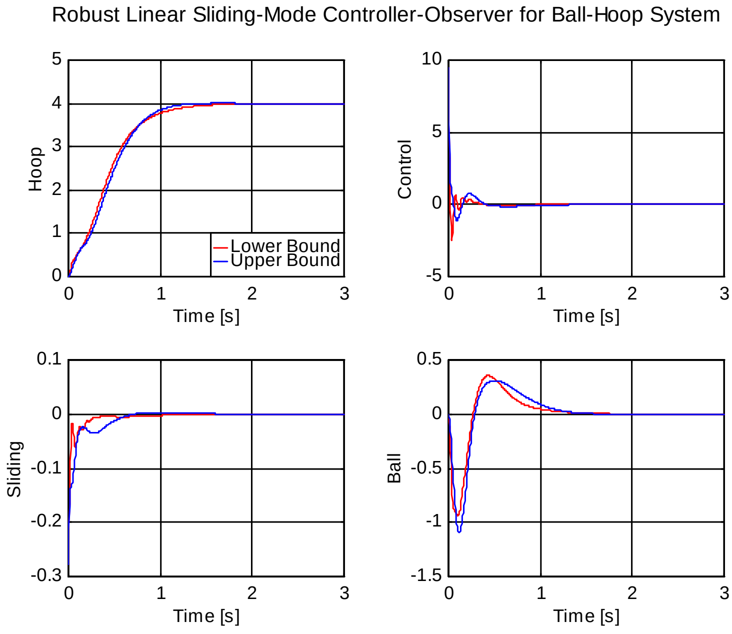
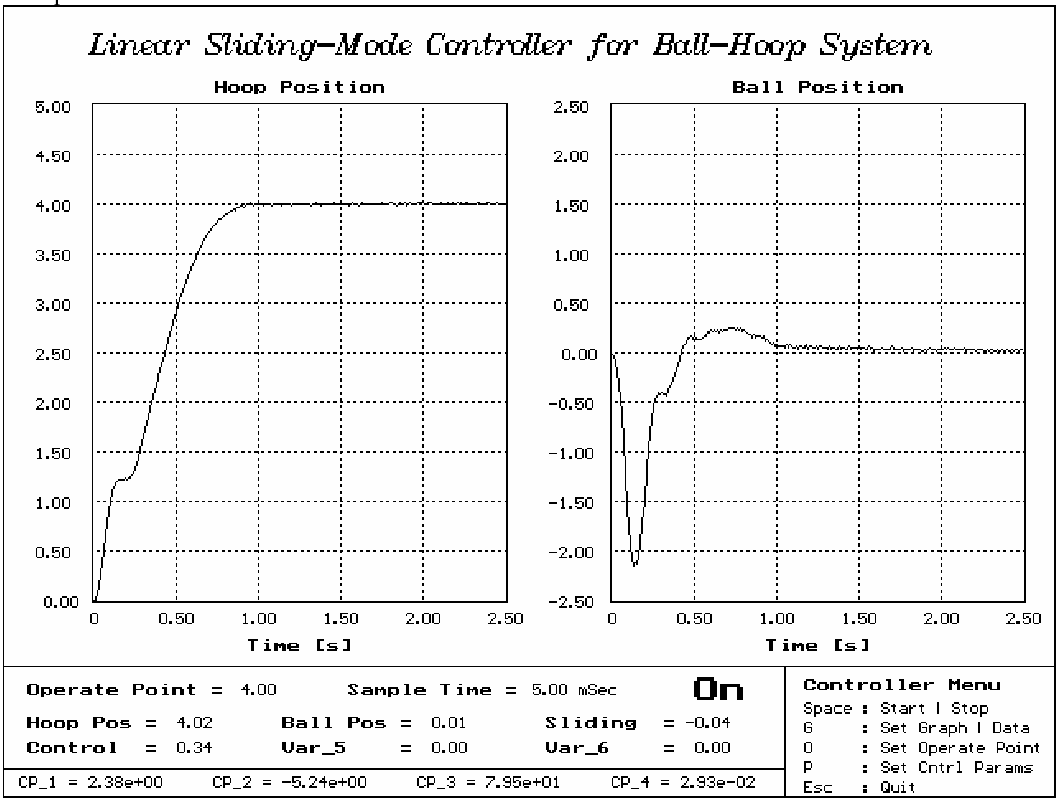


Fig. 8.17: Robust Linear Sliding-Mode Controller-Observer Design for Ball-Hoop System

and the experimental results are



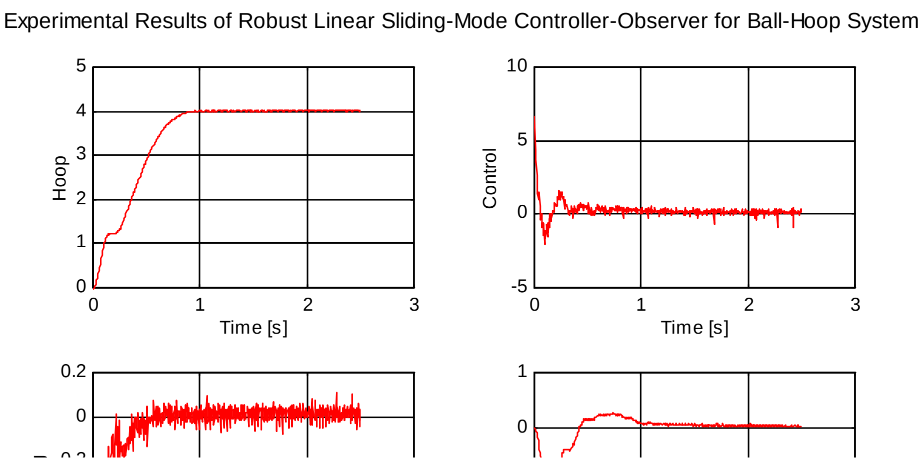


Fig. 8.18: Experimental Results of Robust Linear Sliding-Mode Controller-Observer for Ball-Hoop System

Remark 8.4: Experimental Performance of Pseudo-SMC (Unitvector & TanH SMC) and Linear SMC

The system responses are comparable, however the control effort of Pseudo-SMC is smaller.

8.5.2.5. Robust Sliding-Mode Fuzzy Controller

By the design rule proposed in Corollary 6.4, choose

$$\delta = 8, \quad \delta_i = 0.1$$

and choose 5 fuzzy rules with fuzzy parameters as

$$\mathbf{s} = [-1.5 \quad -0.375 \quad 0 \quad 0.375 \quad 1.5], \quad \mathbf{s}_i = [-1.5 \quad -0.375 \quad 0 \quad 0.375 \quad 1.5], \quad \mathbf{u} = [-10 \quad -5 \quad 0 \quad 5 \quad 10]$$

then the simulation results are

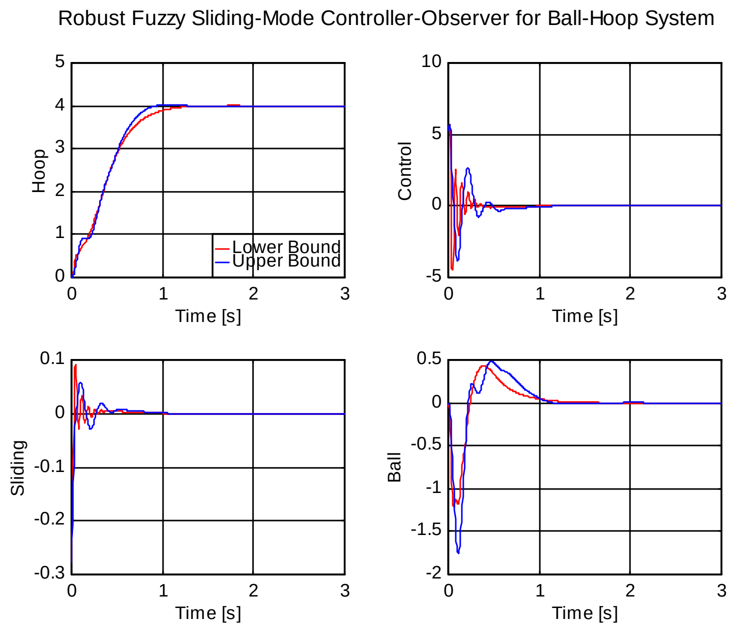
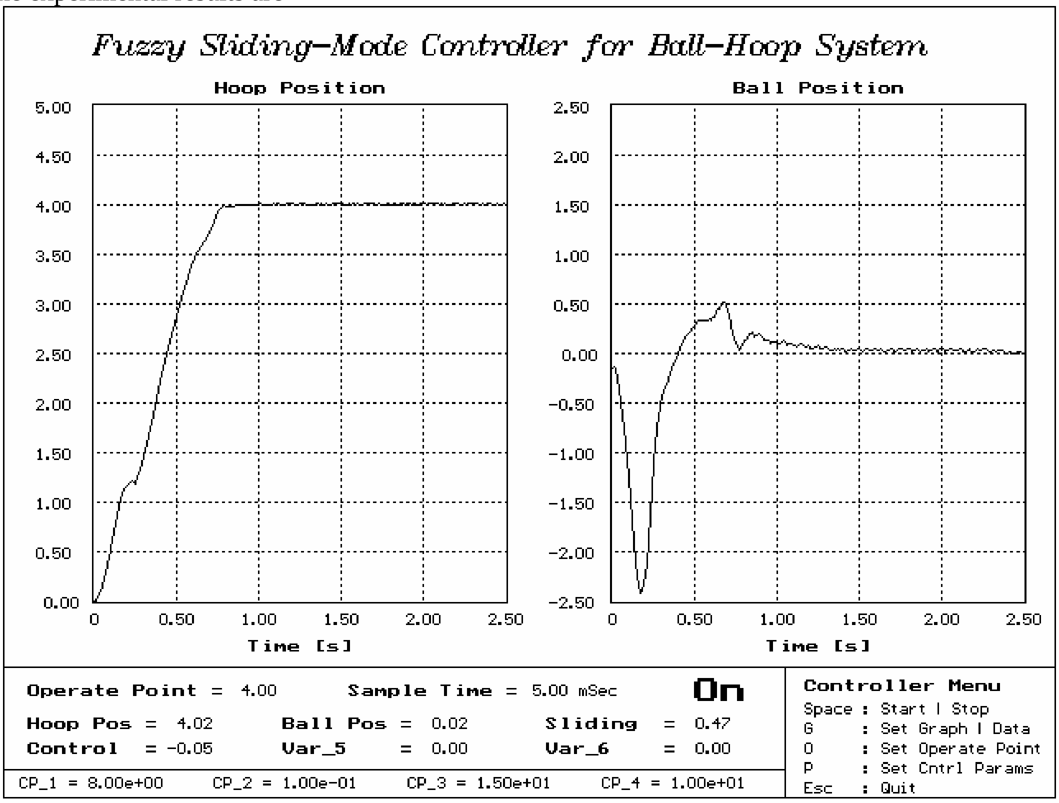


Fig. 8.19: Robust Fuzzy Sliding-Mode Controller-Observer Design for Ball-Hoop System

Remark 8.5: Execute Time of Digital Controllers

The execute time of the sliding-mode fuzzy controller is about 1.17 mS using Turbo Pascal 7 on a PC 486DX-66, while that of others is about 1.1 mS. The controller can be thus completed within 1 sample of 5 mS.

and the experimental results are



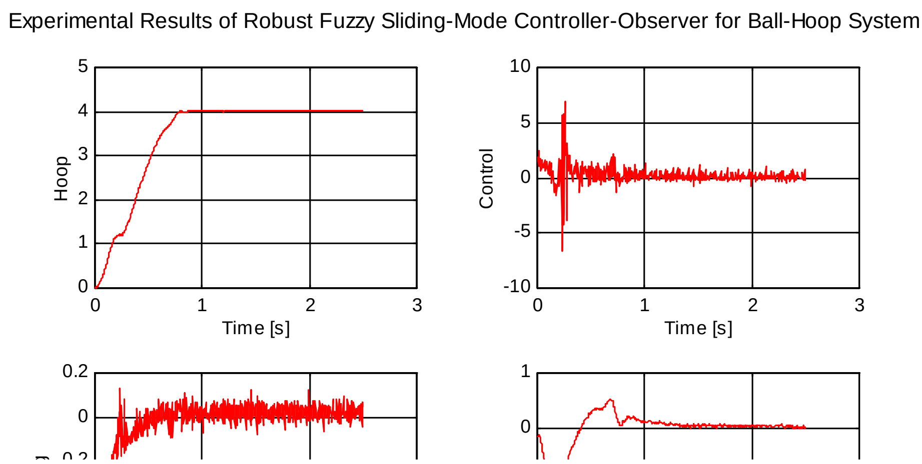


Fig. 8.20: Experimental Results of Robust Fuzzy Sliding-Mode Controller-Observer for Ball-Hoop System

8.6. CONCLUSION

By this experiment we can see the real danger of the switching SMC where it has excited the unmodelled high-frequency. Note that there are some oscillations in the switching SMC controller while it is not the case for other SMC controllers. The limitation of Saturate SMC is its low gain to slow down the response due to the longer reaching mode, as expected in theory. We have attempted with both the saturate and TanH SMC to reduce the gain from infinite gain of the switching SMC. As expected, the performance of the TanH SMC is better than that of the saturate SMC because for the TanH SMC, the higher sliding margin is, the lower gain is; so the oscillation is harder to exist and there is no steady-state error. In other words, the saturate SMC has the upper bound for the sliding margin low enough to slow down the response even with the same fast eigenvalues as in the TanH and linear SMC.

All the experimental results are consistent with the proposed theoretical results, such as the excitation of unmodelled high-frequency by a discontinuous SMC control function, the performances of a saturate and TanH SMC control functions (steady-state error, slow-down response), the performances of a saturate SMC, a TanH SMC and a linear continuous SMC in eliminating the chattering problem.

Chapter 8:

Sliding-Mode Controller: Experimental Results

Advanced Sliding-Mode Controller Designs: Conclusion

In the sliding-mode control theory, control dynamics have 2 sequential modes, the first is the *reaching mode* and the second is the *sliding mode*. The Lyapunov sliding condition forces system states to *reach* a hyperplane and keeps them *sliding* on this hyperplane, so a SMC design is composed of 2 phases, hyperplane design and controller design. First, a hyperplane is designed via the pole-placement approach as in the state-space control, then a controller design is based on the sliding condition. The stability is guaranteed by the sliding condition (Lyapunov Stability Criterion Theorem) *and* by a *stable* hyperplane (stable designer-chosen pole-placement). In the reaching mode, the control dynamics depend on system parameters; but in the sliding mode they depend on the hyperplane, this is the *invariance* property of the sliding mode. On this basis, the design rule (Section 3.4) has been proposed to guarantee the reaching mode terminate in a finite time for the existence of the sliding mode. In addition, the modified design (Corollary 3.2) has been presented to include system dynamics into the reaching control to reinforce this guarantee.

We have identified the sliding-eigenvalues and the hyperplane-eigenvalues. These 2 eigenvalue types are of reduced-order for both linear and nonlinear systems. The sliding-eigenvalues determine the system dynamics in the sliding mode. The hyperplane-eigenvalues are the desired sliding-eigenvalues for a sliding hyperplane of a certain model which is usually chosen to be the nominal one. In other words, the hyperplane-eigenvalues represent the desired sliding plane on which the state variables are expected to slide.

The stability criterion proposed in Section 3.2 is much simpler for practical tests than the works in Utkin 1992, Itkis 1983. This stability approach has been extended to nonlinear systems in Corollary 7.1. To complete the work in Drazenov 1969, a sufficient invariance condition has been presented in Theorem 4.1 and 4.2.

For the discontinuous SMC well-known as VSS, a new reaching controller design in Corollary 3.2 includes the system dynamics to reinforce the guarantee for the existence of the sliding mode to enhance the invariance property. The proposed robust VSS design approach in Section 4.3 is much simpler than the current approach (Edwards *et al.* 1996) and applicable for MIMO uncertain nonlinear systems in Section 7.3.2 (Fu 1992).

The main limitation of VSS is the chattering problem. The analysis of the chattering problem presented in Section 3.5 is more precise than the work in Slotine 1983, its development to use the saturate and unitvector functions is in a unified manner (Saturate function in Slotine 1983, Unitvector function in Ryan *et al.* 1987, Spurgeon 1993) and much simpler than the work in Ryan *et al.* 1987 and Spurgeon

1993. The hyperbolic tangent function has been proposed to solve the chattering problem more convenient since it is a standard mathematical function. Its performance is better than those of saturate and unitvector function. The performances of sliding functions (saturate, unitvector and TanH functions) have been analyzed in detail (Section 3.5.5) and consistent with experimental results in Chapter 8. A steady-state error may occur in using a sliding function to eliminate the chattering, Proposition 3.3 completely solves the chattering problem and its consequence (steady-state error).

The proposed analysis of continuous SMC has been used to explore the true nature of SMC including the invariance property and the stability problem. The control function is not only continuous but also linear. The continuous nature of this control function helps to eliminate the chattering problem in the discontinuous SMC. The advantage of the control function being linear is that some fundamental concepts of SMC design can be explained from the linear control theory framework. In the current SMC literature, a continuous SMC has been presented in DeCarlo *et al.* 1987 and Zhou *et al.* 1992, but the robust continuous SMC has been considered only in Zhou *et al.* 1992. The followings are some main new features compared to the work in Zhou *et al.* 1992:

- the negative of a sliding margin and the hyperplane-eigenvalues are the closed-loop system eigenvalues for linear systems under no perturbation. This concludes the fact that a sliding margin plays a key role in the reaching mode. If this absolute value is larger, the sliding mode is more dominant;
- under matching perturbations, the negative of a sliding margin and the hyperplane-eigenvalues are the closed-loop system eigenvalues for linear systems at one boundary. This may be used to estimate the control performance;
- the control function is linear continuous for linear systems with or without perturbations, and for a certain class of nonlinear systems. In the SMC literature, the control function is linear continuous for linear system without perturbation, but it is non-linear for linear systems under perturbations;

For the fuzzy control, we have proved that a typical fuzzy rulebase can satisfy the Lyapunov sliding condition so the stability of a fuzzy control is guaranteed by the Lyapunov stability theorem. This is a stability criterion for the fuzzy control theory. We have presented a proposition for a fuzzy control structure applicable to slow and fast systems.

To design a stable sliding-mode fuzzy controller, a fuzzy mechanism is used to minimize a sliding variable s instead of using the sliding condition as in the sliding mode control, so we can obtain the invariance property of the sliding mode. In a typical fuzzy rulebase, it may not be convenient to use more than 2 entries, we can use 1 entry for s and the other for sum of s, and hence a possible steady-state error may be eliminated by this I-action.

In a fuzzy control, the problems are how to choose the gains for error and its change; and a possible chattering (limit cycle). In the current fuzzy control literature, these gains are chosen by trial and error or chosen unity without justification, and the unit circle is used to analyze the chattering, not to solve this problem. Using the sliding-mode control theory, these gains can be determined by a hyperplane and the chattering problem can be solved since the system dynamics are included through these gains.

On the basis of the fuzzy identification in Tanaka *et al.* 1992 and Ishigame *et al.* 1993, we develop a new fuzzy identification scheme which is simpler and more practical. The fuzzy inference will be used to obtain the most potential model from some rough mathematical models from experiments using a proposed practical system identification. Due to the robustness, a rough system model is required rather than an elaborate mathematical model as in a conventional control, a practical system identification is presented for this purpose. A fuzzy model by the proposed scheme can be a solution to the conservative problem.

For a general class of uncertain nonlinear systems, a continuous pseudo-SMC may be used to eliminate the chattering problem. In fact, this continuous pseudo-SMC is a discontinuous SMC design where a switching function is replaced by a sliding function. Strictly speaking, a pseudo-SMC is not the sliding mode within a boundary layer, but the fact that the sliding mode exists asymptotically, so the difference between a SMC and a pseudo-SMC is not noticeable.

A general case in SMC has been developed where an output is a nonlinear function of state variables for multi-input multi-output (MIMO) systems. A hyperplane has been derived from the direct allocation method rather than from the input/output linearization technique (Hunt *et al.* 1983, Isidori 1985, Kravaris *et al.* 1986) as in the SMC literature (Fernandez *et al.* 1987; Chen *et al.* 1992). Since the SMC can deal with nonlinear systems conveniently, tracking control problems can be solved without difficulties. For a MIMO SMC, the hierarchical control technique has been used in Utkin 1977 for linear systems. Alternatively, we have used a decoupling technique which is applicable for MIMO nonlinear systems. This technique allows a MIMO can be considered as a collection of SISO subsystem, therefore all our proposed SISO results have been applicable. This technique has been applicable for a certain class of uncertain dynamical MIMO systems (section 7.7.6).

All the experimental results are consistent with the proposed theoretical results

- the excitation of unmodelled high-frequency by a discontinuous SMC control function;
- the performances of a saturate ,unitvector, TanH and linear SMC in eliminating the chattering problem in terms of steady-state error, slow-down response. A linear continuous SMC is free from these 2 disadvantages;

Appendix

A.1. EIGENVALUES, AND SIMILARITY TRANSFORMATION

A.1.1. Eigenvalues of a square matrix

$$\mathrm{eig}(\mathbf{A}) = \{\lambda_i \| \lambda\mathbf{I} - \mathbf{A}| = 0\}, \quad \mathbf{A} \in \Re^{n\times n} \tag{1.1}$$

A.1.2. Similarity Transformation

The matrices $\mathbf{A}, \mathbf{B} \in \Re^{n\times n}$ are said to be *similar* if a non-singular matrix **P** exists such that

$$\mathbf{P}^{-1}\mathbf{A}.\mathbf{P} = \mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{P}.\mathbf{B}\mathbf{P}^{-1} \tag{1.2}$$

The eigenvalues are invariant under a similarity transformation, suppose that

$$\mathbf{P}^{-1}\mathbf{A}.\mathbf{P} = \mathbf{A}$$

then

$$|\mathbf{B}| = |\mathbf{P}^{-1}\mathbf{A}.\mathbf{P}| = |\mathbf{P}^{-1}|.|\mathbf{A}|.|\mathbf{P}| = |\mathbf{A}|.|\mathbf{P}^{-1}|.|\mathbf{P}| = |\mathbf{A}|.|\mathbf{P}^{-1}\mathbf{P}| = |\mathbf{A}|.|\mathbf{I}| = |\mathbf{A}|$$

and

$$|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}.\mathbf{P}| = |\mathbf{P}^{-1}(\lambda\mathbf{I}).\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}.\mathbf{P}| = |\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A}).\mathbf{P}| = |\mathbf{P}^{-1}|.|\lambda\mathbf{I} - \mathbf{A}|.|\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$$

A.2. POLE-PLACEMENT METHOD

A.2.1. Problem Statement

Consider a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}.u \tag{2.1}$$

under a control function

$$u = -\mathbf{K}\mathbf{x} \tag{2.2}$$

then

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}).\mathbf{x} \tag{2.3}$$

where *desired closed-loop poles* are

$$\mathrm{eig}[\mathbf{A} - \mathbf{B}\mathbf{K}] = \mathbf{P} = [\, p_1,\, p_2,\, \cdots,\, p_n \,] \tag{2.4}$$

The problem is to find **K** satisfying Eq.(2.4). We define a procedure **place** as follows

$$\left.\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{K}).\mathbf{x} \\ \mathrm{eig}[\mathbf{A} - \mathbf{B}\mathbf{K}] &= \mathbf{P} \end{aligned}\right\} \Rightarrow \mathbf{K} = \mathrm{place}(\mathbf{A}, \mathbf{B}, \mathbf{P}) \tag{2.5}$$

where this procedure can be determined by a couple of techniques in the literature. Among them is the Ackermann' formula which is the simplest one and applicable to SISO systems only. To illustrate the procedure **place** above, the following is a derivation of the Ackermann's formula.

A.**2.2. Ackermann's Formula** (Ogata 1987)

Let

$$\mathbf{A}_f = \mathbf{A} - \mathbf{BK} \tag{2.6}$$

Since the Caley-Hamilton Theorem states that $\mathbf{A}_f$ satisfies its own characteristic equation, we have

$$\mathbf{A}_f^n + \alpha_1 \mathbf{A}_f^{n-1} + \alpha_2 \mathbf{A}_f^{n-2} + \cdots + \alpha_{n-1}\mathbf{A}_f + \alpha_n \mathbf{I} = \mathscr{P}(\mathbf{A}_f) = \mathbf{0} \tag{2.7}$$

hence

$$\mathbf{I} = \mathbf{I}$$

$$\mathbf{A}_f = \mathbf{A} - \mathbf{BK}$$

$$\mathbf{A}_f^2 = (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKA}_f$$

$$\mathbf{A}_f^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABKA}_f - \mathbf{BKA}_f^2$$

$$\mathbf{A}_f^4 = (\mathbf{A} - \mathbf{BK})^4 = \mathbf{A}^4 - \mathbf{A}^3\mathbf{BK} - \mathbf{A}^2\mathbf{BKA}_f - \mathbf{ABKA}_f^2 - \mathbf{BKA}_f^3$$

$$\mathbf{A}_f^5 = (\mathbf{A} - \mathbf{BK})^5 = \mathbf{A}^5 - \mathbf{A}^4\mathbf{BK} - \mathbf{A}^3\mathbf{BKA}_f - \mathbf{A}^2\mathbf{BKA}_f^2 - \mathbf{ABKA}_f^3 - \mathbf{BKA}_f^4$$

$$\cdot \quad \cdot \quad \cdot$$

$$\mathbf{A}_f^n = (\mathbf{A} - \mathbf{BK})^n = \mathbf{A}^n - \mathbf{A}^{n-1}\mathbf{BK} - \cdots - \mathbf{BKA}_f^{n-1}$$

Multiply by $\alpha_n, \alpha_{n-1}, \cdots, \alpha_1, 1$, respectively:

$$\alpha_n\mathbf{I} + \alpha_{n-1}\mathbf{A}_f + \alpha_{n-2}\mathbf{A}_f^2 + \cdots + \mathbf{A}_f^n = \alpha_n\mathbf{I} + \alpha_{n-1}\mathbf{A} + \alpha_{n-2}\mathbf{A}^2 + \alpha_{n-3}\mathbf{A}^3 + \cdots + \mathbf{A}^n - \alpha_{n-1}\mathbf{BK} - \alpha_{n-2}\mathbf{ABK} - \alpha_{n-2}\mathbf{BKA}_f$$
$$- \alpha_{n-3}\mathbf{A}^2\mathbf{BK} - \alpha_{n-3}\mathbf{ABKA}_f - \alpha_{n-3}\mathbf{BKA}_f^2 - \cdots - \mathbf{A}^{n-1}\mathbf{BK} - \cdots - \mathbf{BKA}_f^{n-1}$$

or equivalently

$$\mathscr{P}(\mathbf{A}_f) = \mathscr{P}(\mathbf{A}) - \alpha_{n-1}(\mathbf{B})\mathbf{K} - \alpha_{n-2}(\mathbf{AB})\mathbf{K} - \alpha_{n-2}(\mathbf{B})\mathbf{KA}_f - \alpha_{n-3}(\mathbf{A}^2\mathbf{B})\mathbf{K} - \alpha_{n-3}(\mathbf{AB})\mathbf{KA}_f - \alpha_{n-3}(\mathbf{B})\mathbf{KA}_f^2 - \cdots - (\mathbf{A}^{n-1}\mathbf{B})\mathbf{K} - \cdots - (\mathbf{B})\mathbf{KA}_f^{n-1}$$

$$\mathscr{P}(\mathbf{A}_f) = \mathscr{P}(\mathbf{A}) - \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} . \begin{bmatrix} \alpha_{n-1}\mathbf{K} + \alpha_{n-2}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-1} \\ \alpha_{n-2}\mathbf{K} + \alpha_{n-3}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-2} \\ \cdots \\ \mathbf{K} \end{bmatrix}$$

with

$$\mathscr{P}(\mathbf{A}_f) = \mathbf{0} \tag{2.8}$$

and

$$\mathbf{M}_c = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} : \textit{Controllability Matrix} \tag{2.9}$$

then

$$\mathscr{P}(\mathbf{A}) = \mathbf{M}_c \begin{bmatrix} \alpha_{n-1}\mathbf{K} + \alpha_{n-2}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-1} \\ \alpha_{n-2}\mathbf{K} + \alpha_{n-3}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-2} \\ \cdots \\ \mathbf{K} \end{bmatrix} \tag{2.10}$$

if $\mathbf{M}_c$ is *invertible* (*controllable*), then

$$\begin{bmatrix} \alpha_{n-1}\mathbf{K} + \alpha_{n-2}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-1} \\ \alpha_{n-2}\mathbf{K} + \alpha_{n-3}\mathbf{KA}_f + \cdots + \mathbf{KA}_f^{n-2} \\ \cdots \\ \mathbf{K} \end{bmatrix} = \mathbf{M}_c^{-1} . \mathscr{P}(\mathbf{A}) \tag{2.11}$$

Premultiplying both side by $\left[0, 0, \cdots, 0, 1\right]$

$$\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}.\begin{bmatrix}\alpha_{n-1}\mathbf{K}+\alpha_{n-2}\mathbf{KA}_f+\cdots+\mathbf{KA}_f^{n-1}\\ \alpha_{n-2}\mathbf{K}+\alpha_{n-3}\mathbf{KA}_f+\cdots+\mathbf{KA}_f^{n-2}\\ \cdots \\ \mathbf{K}\end{bmatrix}=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}.\mathbf{M}_c^{-1}.\mathscr{P}(\mathbf{A}) \qquad (2.12)$$

Therefore, the procedure **place** in Eq.(2.5) calculates **K** by the Ackermann's formula as follows

$$\mathbf{K}=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\mathbf{M}_c^{-1}.\mathscr{P}(\mathbf{A}) \qquad (2.13)$$

A.3. MATRIX OPERATIONS

A.3.1. Differentiation and Derivative of a Matrix

A.3.1.1. Differentiation

$$d(\mathbf{A}+\mathbf{B})=d\mathbf{A}+d\mathbf{B} \qquad (3.1)$$

$$d(\mathbf{AB})=d\mathbf{A}.\mathbf{B}+\mathbf{A}.d\mathbf{B} \qquad (3.2)$$

Eq.(3.2) gives

$$\mathbf{0}=d\mathbf{I}=d\left(\mathbf{A}.\mathbf{A}^{-1}\right)=d\mathbf{A}.\mathbf{A}^{-1}+\mathbf{A}.d\left(\mathbf{A}^{-1}\right)$$

thus

$$d\left(\mathbf{A}^{-1}\right)=-\mathbf{A}^{-1}.d\mathbf{A}.\mathbf{A}^{-1} \qquad (3.3)$$

A.3.1.2. Derivative

$$\frac{d}{dt}V\left(\mathbf{x}(t)\right)=\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T\frac{d\mathbf{x}}{dt} \qquad (3.4)$$

GRADIENT

$$\nabla V(\mathbf{x})=\frac{\partial V}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial V}{\partial x_1}\\ \vdots \\ \frac{\partial V}{\partial x_n}\end{bmatrix}$$

HESSIAN

$$\nabla^2 V(\mathbf{x})=\frac{\partial^2 V}{\partial \mathbf{x}^2}=\begin{bmatrix}\frac{\partial^2 V}{\partial x_1\partial x_1}\times\frac{\partial^2 V}{\partial x_1\partial x_2}\times\cdots\times\frac{\partial^2 V}{\partial x_1\partial x_n}\\ \vdots \\ \frac{\partial^2 V}{\partial x_n\partial x_1}\times\frac{\partial^2 V}{\partial x_n\partial x_2}\times\cdots\times\frac{\partial^2 V}{\partial x_n\partial x_n}\end{bmatrix}$$

A.3.1.3. Jacobian

$$\nabla\mathbf{f}=\frac{\partial \mathbf{f}}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1}\\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix} \qquad \mathbf{(3.5)}$$

$$\frac{\partial}{\partial \mathbf{x}}\mathbf{Ax}=\mathbf{A}^T, \qquad \frac{\partial}{\partial \mathbf{x}}\mathbf{x}^T\mathbf{Ax}=\mathbf{Ax}+\mathbf{A}^T\mathbf{x}, \qquad \frac{\partial}{\partial \mathbf{x}}\mathbf{x}^T\mathbf{Ay}=\mathbf{Ay}, \qquad \frac{\partial}{\partial \mathbf{y}}\mathbf{x}^T\mathbf{Ay}=\mathbf{A}^T\mathbf{x}$$

A.3.2. Trace of an *n*×*n* Matrix

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} \tag{3.6}$$

(i) $$\mathrm{tr}\left(\mathbf{A}^T\right) = \mathrm{tr}(\mathbf{A}), \qquad \{\mathrm{tr}(\mathbf{A})\}^T = \mathrm{tr}(\mathbf{A}) \tag{3.7}$$

(ii) $$\mathrm{tr}(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B}) \tag{3.8}$$

(iii) $$\mathrm{tr}(\mathbf{AB}) = tr(\mathbf{BA}) \tag{3.9}$$

(iv) $$\mathrm{tr}\left(\mathbf{M}^{-1}\mathbf{AM}\right) = \mathrm{tr}(\mathbf{A}) \tag{3.10}$$

(v) $$\mathbf{x}^T\mathbf{Qx} = \mathrm{tr}\left(\mathbf{xx}^T\mathbf{Q}\right) = \mathrm{tr}\left(\mathbf{Qxx}^T\right) \tag{3.11}$$

(vi) $$\mathrm{tr}(\mathbf{AX}) = \mathrm{tr}(\mathbf{XA}) = \mathrm{tr}\left(\mathbf{A}^T\mathbf{X}^T\right) = \mathrm{tr}\left(\mathbf{X}^T\mathbf{A}^T\right) \tag{3.12}$$

Trace operator is unchanged under : swap, transpose (each individually).

A.3.3. The Caley-Hamilton theorem

Let **A** be square matrix, and let

$$|s\mathbf{I} - \mathbf{A}| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \tag{3.13}$$

then

$$\mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_n\mathbf{I} = \mathbf{0} \tag{3.14}$$

A.3.4. Inversion of Matrices

A.3.4.1. Nonsingular Matrix and Singular Matrix

A square matrix **A** is called a nonsingular matrix if a matrix **B** exists such that

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$$

If such a matrix **B** exists, then it is denoted by $\mathbf{A}^{-1}$. $\mathbf{A}^{-1}$ is called the *inverse* of **A**. The inverse matrix $\mathbf{A}^{-1}$ exists if $|\mathbf{A}|$ is nonzero. If $\mathbf{A}^{-1}$ does not exist, **A** is said to be *singular*.

If **A** and **B** are nonsingular matrices, then the product **AB** is a nonsingular matrix and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{3.15}$$

Also

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T \tag{3.16}$$

For a 2×2 matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad |\mathbf{A}| = ad - bc \neq 0$$

the inverse matrix is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = |\mathbf{A}|^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A.3.4.2. Matrix Inversion Lemma

If **A, B, C** and **D** are, respectively, an $n \times n$, an $n \times m$, an $m \times m$ and an $m \times n$ matrix, then

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1}+\mathbf{DA}^{-1}\mathbf{B}\right)^{-1}\mathbf{DA}^{-1} \tag{3.17}$$

provided the indicated inverses exist.

A.3.4.3. Block Matrix Inversion

If **A, B, C** and **D** are, respectively, an $n \times n$, an $n \times m$, an $m \times n$ and an $m \times m$ matrix, then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1}, & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1}, & \left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1} \end{bmatrix} \tag{3.18}$$

provided $|\mathbf{A}| \neq 0$ and $\left|\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right| \neq 0$, or

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \left(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}\right)^{-1}, & -\left(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}\right)^{-1}\mathbf{BD}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}\right)^{-1}, & \mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}\right)^{-1}\mathbf{BD}^{-1}+\mathbf{D}^{-1} \end{bmatrix} \tag{3.19}$$

provided $|\mathbf{D}| \neq 0$ and $\left|\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}\right| \neq 0$

In particular, if $\mathbf{C}=\mathbf{0}$ or $\mathbf{B}=\mathbf{0}$, then Eqs.(3.18) and (3.19) can be simplified as follows

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}, & -\mathbf{A}^{-1}\mathbf{BD}^{-1} \\ \mathbf{0}, & \mathbf{D}^{-1} \end{bmatrix} \tag{3.20}$$

or

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}, & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{CA}^{-1}, & \mathbf{D}^{-1} \end{bmatrix} \tag{3.21}$$

A.3.5. Gradient Matrices and Frechet Derivative Operator. (H. P. Geering, AC-21/8, 1976)

A.3.5.1. Frechet Derivative Operator.

Consider a scalar-valued function $f(\mathbf{X}): R^{n\times m} \to R$

$$f(\mathbf{X}) = \mathrm{tr}\left(\mathbf{A}^T\mathbf{X}\right)$$

then the First Frechet derivative at **X** with increment $d\mathbf{X}$ is

$$df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}\left(\mathbf{A}^T d\mathbf{X}\right) = \mathrm{tr}\left(d\mathbf{X}^T\mathbf{A}\right)$$

and the First Frechet derivative at **X** with respect to **X** is

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = f'(\mathbf{X}) = \left[\frac{\partial f(\mathbf{X})}{\partial x_{ij}}\right] = \mathbf{A}$$

A.3.5.2. Rule for Differentiating Functions Involving the Trace Operator

Consider $f: R^{n\times m} \to R$ of the form $f(\mathbf{X}) = \mathrm{tr}\left[g(\mathbf{X})\right]$, where $g: R^{n\times m} \to R^{r\times r}$, g is a matrix polynomial in $\mathbf{X}$. The First Frechet derivative at $\mathbf{X}$ of $f(\mathbf{X})$ is obtained as followed

(1) Differentiate separately each term of the polynomial

(2) For each term, form the partial Frechet differential for each factor $\mathbf{X}$ separately by placing it by $d\mathbf{X}$. To each partial differential, apply Id.(*vi*) until $d\mathbf{X}^T$ appears at the extreme left. Hence, each partial differential is of the form $\mathrm{tr}\left(d\mathbf{X}^T\mathbf{M}\right)$ for some $\mathbf{M} \in R^{n\times m}$. The partial derivative corresponding to this partial differential is $\mathbf{M}$.

(3) The derivative $f'(\mathbf{X})$ is obtained by summing over all of the partial derivatives of all of the terms of the polynomial.

A.3.6 Some Gradient Matrices. (M. Athans, Inf. Ctl. 11, 1968)

The followings are derivations of some gradient matrices in Athans 1968 by using the above Frechet derivative operator

(1)
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}[\mathbf{X}] = \mathbf{I} \Rightarrow f(\mathbf{X}) = \mathrm{tr}[\mathbf{X}]$$
$$df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}[d\mathbf{X}] = \mathrm{tr}\left[d\mathbf{X}^T\mathbf{I}\right] \Rightarrow f'(\mathbf{X}) = \mathbf{I}$$

(2)
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}[\mathbf{AX}] = \mathbf{A}^T \Rightarrow f(\mathbf{X}) = \mathrm{tr}[\mathbf{AX}]$$
$$df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}[\mathbf{A}.d\mathbf{X}] = \mathrm{tr}\left[d\mathbf{X}^T\mathbf{A}^T\right] \Rightarrow f'(\mathbf{X}) = \mathbf{A}^T$$

(3)
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}\left[\mathbf{AX}^T\right] = \mathbf{A} \Rightarrow f(\mathbf{X}) = \mathrm{tr}\left[\mathbf{AX}^T\right]$$
$$df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}\left[\mathbf{A}.d\mathbf{X}^T\right] = \mathrm{tr}\left[d\mathbf{X}^T\mathbf{A}\right] \Rightarrow f'(\mathbf{X}) = \mathbf{A}$$

(4)
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}[\mathbf{AXB}] = \mathbf{A}^T\mathbf{B}^T$$
$$f(\mathbf{X}) = \mathrm{tr}[\mathbf{AXB}] \Rightarrow df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}[(\mathbf{A}.d\mathbf{X}).\mathbf{B}] = \mathrm{tr}\left[d\mathbf{X}^T\mathbf{A}^T.\mathbf{B}^T\right] \Rightarrow f'(\mathbf{X}) = \mathbf{A}^T.\mathbf{B}^T$$

(5)
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}\left[\mathbf{AX}^T\mathbf{B}\right] = \mathbf{BA} \Rightarrow f(\mathbf{X}) = \mathrm{tr}\left[\mathbf{AX}^T\mathbf{B}\right]$$
$$df(\mathbf{X}, d\mathbf{X}) = \mathrm{tr}\left[\left(\mathbf{A}.d\mathbf{X}^T\right).\mathbf{B}\right] = \mathrm{tr}\left[\mathbf{B}.\left(\mathbf{A}.d\mathbf{X}^T\right)\right] = \mathrm{tr}\left[(\mathbf{B}.\mathbf{A}).d\mathbf{X}^T\right] = \mathrm{tr}\left[d\mathbf{X}^T(\mathbf{B}.\mathbf{A})\right] \Rightarrow f'(\mathbf{X}) = \mathbf{B}.\mathbf{A}$$

(6)
$$\frac{\partial}{\partial \mathbf{X}^T}\mathrm{tr}[\mathbf{AX}] = \mathbf{A}$$
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}[\mathbf{AX}] = \mathbf{A}^T \Rightarrow \frac{\partial}{\partial \mathbf{X}^T}\mathrm{tr}[\mathbf{AX}] = \mathbf{A}$$

(7)
$$\frac{\partial}{\partial \mathbf{X}^T}\mathrm{tr}\left[\mathbf{AX}^T\right] = \mathbf{A}^T$$
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}\left[\mathbf{AX}^T\right] = \mathbf{A} \Rightarrow \frac{\partial}{\partial \mathbf{X}^T}\mathrm{tr}\left[\mathbf{AX}^T\right] = \mathbf{A}^T$$

(8)
$$\frac{\partial}{\partial \mathbf{X}^T}\operatorname{tr}[\mathbf{AXB}] = \mathbf{BA}$$
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{AXB}] = \mathbf{A}^T\mathbf{B}^T \Rightarrow \frac{\partial}{\partial \mathbf{X}^T}\operatorname{tr}[\mathbf{AXB}] = \mathbf{BA}$$

(9)
$$\frac{\partial}{\partial \mathbf{X}^T}\operatorname{tr}\left[\mathbf{AX}^T\mathbf{B}\right] = \mathbf{A}^T\mathbf{B}^T$$
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{AX}^T\mathbf{B}\right] = \mathbf{BA} \Rightarrow \frac{\partial}{\partial \mathbf{X}^T}\operatorname{tr}\left[\mathbf{AX}^T\mathbf{B}\right] = \mathbf{A}^T\mathbf{B}^T$$

(10)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{XX}] = 2\mathbf{X}^T \Rightarrow f(\mathbf{X}) = \operatorname{tr}[\mathbf{XX}]$$
$$df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}[d\mathbf{X}.\mathbf{X}] + \operatorname{tr}[\mathbf{X}.d\mathbf{X}] = \operatorname{tr}\left[d\mathbf{X}^T.\mathbf{X}^T\right] + \operatorname{tr}\left[d\mathbf{X}^T.\mathbf{X}^T\right]$$
$$f'(\mathbf{X}) = \mathbf{X}^T + \mathbf{X}^T = 2\mathbf{X}^T$$

(11)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{XX}^T\right] = 2\mathbf{X} \Rightarrow f(\mathbf{X}) = \operatorname{tr}\left[\mathbf{XX}^T\right]$$
$$df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}\left[d\mathbf{X}.\mathbf{X}^T\right] + \operatorname{tr}\left[\mathbf{X}.d\mathbf{X}^T\right] = \operatorname{tr}\left[d\mathbf{X}^T.\mathbf{X}\right] + \operatorname{tr}\left[d\mathbf{X}^T.\mathbf{X}\right]$$
$$f'(\mathbf{X}) = \mathbf{X} + \mathbf{X} = 2\mathbf{X}$$

(12)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{X}^n\right] = n\left(\mathbf{X}^{n-1}\right)^T$$

(13)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{A}.\mathbf{X}^n\right] = \left(\sum_{i=0}^{n-1}\mathbf{X}^i.\mathbf{A}.\mathbf{X}^{n-i-1}\right)^T$$

(14)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{AXBX}] = \mathbf{A}^T\mathbf{X}^T\mathbf{B}^T + \mathbf{B}^T\mathbf{X}^T\mathbf{A}^T \Rightarrow f(\mathbf{X}) = \operatorname{tr}[\mathbf{AXBX}]$$
$$df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}[\mathbf{A}.d\mathbf{X}.\mathbf{BX}] + \operatorname{tr}[(\mathbf{AXB}).d\mathbf{X}] = \operatorname{tr}\left[d\mathbf{X}^T\mathbf{A}^T\mathbf{X}^T\mathbf{B}^T\right] + \operatorname{tr}\left[d\mathbf{X}^T\mathbf{B}^T\mathbf{X}^T\mathbf{A}^T\right]$$
$$f'(\mathbf{X}) = \mathbf{A}^T\mathbf{X}^T\mathbf{B}^T + \mathbf{B}^T\mathbf{X}^T\mathbf{A}^T$$

(15)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{AXBX}^T\right] = \mathbf{A}^T\mathbf{XB}^T + \mathbf{AXB} \Rightarrow f(\mathbf{X}) = \operatorname{tr}\left[\mathbf{AXBX}^T\right]$$
$$df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}\left[\mathbf{A}.d\mathbf{X}.\mathbf{BX}^T\right] + \operatorname{tr}\left[\mathbf{AXB}.d\mathbf{X}^T\right] = \operatorname{tr}\left[d\mathbf{X}^T\mathbf{A}^T\mathbf{XB}^T\right] + \operatorname{tr}\left[d\mathbf{X}^T\mathbf{AXB}\right]$$
$$f'(\mathbf{X}) = \mathbf{A}^T\mathbf{XB}^T + \mathbf{AXB}$$

(16)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[e^{\mathbf{X}}\right] = e^{\mathbf{X}^T}$$
$$f(\mathbf{X}) = \operatorname{tr}\left[e^{\mathbf{X}}\right] = \operatorname{tr}\left[\mathbf{I} + \mathbf{X} + \frac{1}{2!}\mathbf{X}^2 + \cdots + \frac{1}{k!}\mathbf{X}^k + \cdots\right]$$
$$df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}\left[\mathbf{0} + d\mathbf{X} + \frac{1}{2!}(d\mathbf{X}.\mathbf{X} + \mathbf{X}.d\mathbf{X}) + \cdots + \frac{1}{k!}\left(d\mathbf{X}.\mathbf{X}^{k-1} + \cdots + \mathbf{X}^{k-1}d\mathbf{X}\right) + \cdots\right] = \operatorname{tr}\left[e^{\mathbf{X}}d\mathbf{X}\right] = \operatorname{tr}\left[d\mathbf{X}^T e^{\mathbf{X}^T}\right]$$

(17)
$$\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}\left[\mathbf{X}^{-1}\right] = -\left(\mathbf{X}^{-1}\mathbf{X}^{-1}\right)^T = -\left(\mathbf{X}^{-2}\right)^T$$
$$d\left(\mathbf{X}^{-1}\mathbf{X}\right) = d(\mathbf{I}) = 0 = d\left(\mathbf{X}^{-1}\right)\mathbf{X} + \mathbf{X}^{-1}d\mathbf{X} \Rightarrow d\left(\mathbf{X}^{-1}\right) = -\mathbf{X}^{-1}d\mathbf{X}.\mathbf{X}^{-1}$$
$$f(\mathbf{X}) = \operatorname{tr}\left[\mathbf{X}^{-1}\right] \Rightarrow df(\mathbf{X}, d\mathbf{X}) = \operatorname{tr}\left[d\left(\mathbf{X}^{-1}\right)\right] = \operatorname{tr}\left[-\mathbf{X}^{-1}d\mathbf{X}.\mathbf{X}^{-1}\right] = -\operatorname{tr}\left[d\mathbf{X}^T\left(\mathbf{X}^{-1}\right)^T\left(\mathbf{X}^{-1}\right)^T\right]$$

(18)
$$\frac{\partial}{\partial \mathbf{X}}\text{tr}\left[\mathbf{A}\mathbf{X}^{-1}\mathbf{B}\right] = -\left(\mathbf{X}^{-1}\mathbf{B}\mathbf{A}\mathbf{X}^{-1}\right)^T \Rightarrow f(\mathbf{X}) = \text{tr}\left[\mathbf{A}\mathbf{X}^{-1}\mathbf{B}\right]$$

$$df(\mathbf{X}, d\mathbf{X}) = \text{tr}\left[\mathbf{A}d\left(\mathbf{X}^{-1}\right)\mathbf{B}\right] = \text{tr}\left[-\mathbf{A}\mathbf{X}^{-1}d\mathbf{X}.\mathbf{X}^{-1}\mathbf{B}\right] = -\text{tr}\left[d\mathbf{X}^T\left(\mathbf{A}\mathbf{X}^{-1}\right)^T.\left(\mathbf{X}^{-1}\mathbf{B}\right)^T\right] = -\text{tr}\left[d\mathbf{X}^T\left(\mathbf{X}^{-1}\mathbf{B}\mathbf{A}\mathbf{X}^{-1}\right)^T\right]$$

$$f'(\mathbf{X}) = \left(\mathbf{X}^{-1}\mathbf{B}\mathbf{A}\mathbf{X}^{-1}\right)^T$$

A.4. SOME PROPERTIES OF A DETERMINANT

The determinant of an $n \times n$ matrix has the following properties:

1. If 2 rows (or 2 columns) of the determinant are interchanged, only the sign of the determinant is changed.

2. The determinant is invariant under the addition of a scalar multiple of a row (or column) to another row (or column).

3. If an $n \times n$ matrix has 2 identical rows (or columns), then the determinant is zero.

4. For an $n \times n$ matrix **A**,
$$\left|\mathbf{A}^T\right| = \left|\mathbf{A}\right|$$

5. The determinant of a product of 2 square matrices **A** and **B** is the product of their determinants:
$$\left|\mathbf{A}.\mathbf{B}\right| = \left|\mathbf{A}\right|.\left|\mathbf{B}\right| = \left|\mathbf{B}.\mathbf{A}\right|$$

6. If a row (or column) is multiplied by *k*, then the determinant is multiplied by *k*.

7. If all elements of an $n \times n$ matrix are multiplied by k^n; that is
$$\left|k.\mathbf{A}\right| = k^n\left|\mathbf{A}\right|$$

8. If the eigenvalues of **A** are $\lambda_i, \quad i \in [1, n]$, then
$$\left|\mathbf{A}\right| = \lambda_1\lambda_2\ldots\lambda_n$$
Hence $\left|\mathbf{A}\right| \neq 0$ implies $\lambda_i \neq 0, \quad \forall i \in [1, n]$

9. If matrices $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{B} \in \Re^{n\times m}$, $\mathbf{C} \in \Re^{m\times n}$, and $\mathbf{D} \in \Re^{m\times m}$, then
$$\begin{vmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{vmatrix} = \begin{cases}\left|\mathbf{A}\right|.\left|\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right|, & \text{if } \left|\mathbf{A}\right| \neq 0\\ \left|\mathbf{D}\right|.\left|\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right|, & \text{if } \left|\mathbf{C}\right| \neq 0\end{cases} \Rightarrow \begin{vmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{D}\end{vmatrix} = \begin{vmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C} & \mathbf{D}\end{vmatrix} = \left|\mathbf{A}\right|.\left|\mathbf{D}\right|$$

10. For an $\mathbf{A} \in \Re^{n\times m}$, $\mathbf{B} \in \Re^{m\times n}$
$$\left|\mathbf{I}_n + \mathbf{A}\mathbf{B}\right| = \left|\mathbf{I}_m + \mathbf{B}\mathbf{A}\right| \quad \Rightarrow \quad \left|\mathbf{I}_n + \mathbf{A}\mathbf{B}\right| = 1 + \mathbf{B}\mathbf{A}, \quad \text{for } m = 1$$

A.5. LIE ALGEBRA. (A. Isidori 2ed. 1989).

A.5.1. Covector Field Differential

The CoVector field **differential,** or **gradient** $d\lambda$: $\lambda(\mathbf{x})$: Real-Valued function

$$d\lambda(\mathbf{x}) = \frac{\partial\lambda}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial\lambda}{\partial x_1} & \frac{\partial\lambda}{\partial x_2} & \cdots & \frac{\partial\lambda}{\partial x_n} \end{bmatrix} \tag{5.1}$$

to be mathematically precise

$$d\lambda(\mathbf{x}) \to \nabla\lambda(\mathbf{x}) \Rightarrow d\lambda(\mathbf{x}) = \nabla\lambda(\mathbf{x}).d\mathbf{x} = \sum \frac{\partial\lambda}{\partial x_i} dx_i$$

thus

$$\nabla\lambda(\mathbf{x}) = \begin{bmatrix} \frac{\partial\lambda}{\partial x_1} & \frac{\partial\lambda}{\partial x_2} & \cdots & \frac{\partial\lambda}{\partial x_n} \end{bmatrix}$$

and

$$\nabla\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

A.5.2. First Type of Differential Operation

Lie Differentiation : Derivative of a real-valued function λ along the vector fields **f**

$$L_{\mathbf{f}}\lambda(\mathbf{x}) = \langle \nabla\lambda(\mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle = \nabla\lambda(\mathbf{x}).\mathbf{f}(\mathbf{x}) = \frac{\partial\lambda}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial\lambda}{\partial x_i} f_i(\mathbf{x}) \tag{5.2}$$

For a repeated differentiating

$$L_{\mathbf{g}}L_{\mathbf{f}}\lambda(\mathbf{x}) = L_{\mathbf{g}}\left(L_{\mathbf{f}}\lambda(\mathbf{x})\right) = \nabla\left(L_{\mathbf{f}}\lambda\right).\mathbf{g}(\mathbf{x}) \tag{5.3}$$

or in a recursion

$$L_{\mathbf{f}}^{k}\lambda(\mathbf{x}) = \nabla\left(L_{\mathbf{f}}^{k-1}\lambda\right).\mathbf{f}(\mathbf{x}) = L_{\mathbf{f}}L_{\mathbf{f}}^{k-1}\lambda(\mathbf{x}) \tag{5.4}$$

where

$$\begin{cases} L_{\mathbf{f}}^{0}\lambda(\mathbf{x}) = \lambda(\mathbf{x}), \qquad L_{\mathbf{f}}^{1}\lambda(\mathbf{x}) = \nabla\left(L_{\mathbf{f}}^{0}\lambda\right).\mathbf{f}(\mathbf{x}) = \nabla\lambda.\mathbf{f}(\mathbf{x}) \\ L_{\mathbf{f}}^{2}\lambda(\mathbf{x}) = \nabla\left(L_{\mathbf{f}}^{1}\lambda\right).\mathbf{f}(\mathbf{x}) = \nabla\left[\nabla\lambda.\mathbf{f}(\mathbf{x})\right].\mathbf{f}(\mathbf{x}) \\ L_{\mathbf{f}}^{3}\lambda(\mathbf{x}) = \nabla\left(L_{\mathbf{f}}^{2}\lambda\right).\mathbf{f}(\mathbf{x}) = \nabla\left[\nabla\left[\nabla\lambda.\mathbf{f}(\mathbf{x})\right].\mathbf{f}(\mathbf{x})\right].\mathbf{f}(\mathbf{x}) \\ \cdots \end{cases}$$

A.5.3. Second Type Differential Operation

Derivative of a <u>CoVector field</u> along a Vector field, compare to (2) for a Scalar instead of a Vector.

$$L_{\mathbf{f}}\omega(\mathbf{x}) = \mathbf{f}^{T}(\mathbf{x}).\left[\nabla\omega^{T}\right]^{T} + \omega(\mathbf{x}).\nabla\mathbf{f} \tag{5.5}$$

A.5.4. Third Type of Differential Operation

Lie bracket (Lie product) of both Vector fields **f,g**

$$[\mathbf{f},\mathbf{g}](\mathbf{x}) = \nabla\mathbf{g}.\mathbf{f}(\mathbf{x}) - \nabla\mathbf{f}.\mathbf{g}(\mathbf{x}) \tag{5.6}$$

for a repeated bracketing $[\mathbf{f},[\mathbf{f},\ldots,[\mathbf{f},\mathbf{g}]]]$, to avoid by the following definition:

$$ad_{\mathbf{f}}^{k}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{k-1}\mathbf{g}](\mathbf{x}), \quad k \ge 1 \tag{5.7}$$

$$\left.\begin{aligned}
&ad_{\mathbf{f}}^{0}\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x});\\
&ad_{\mathbf{f}}^{1}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{0}\mathbf{g}] = [\mathbf{f},\mathbf{g}]\\
&ad_{\mathbf{f}}^{2}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{1}\mathbf{g}] = [\mathbf{f},[\mathbf{f},\mathbf{g}]]\\
&ad_{\mathbf{f}}^{3}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{2}\mathbf{g}] = [\mathbf{f},[\mathbf{f},[\mathbf{f},\mathbf{g}]]]\\
&\ldots\\
&ad_{\mathbf{f}}^{k}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{k-1}\mathbf{g}] = [\mathbf{f},[\mathbf{f},\ldots,[\mathbf{f},\mathbf{g}]]]
\end{aligned}\right\} \tag{5.8}$$

A.5.5. Lie algebra

(1) Bilinear : $r_1, r_2 \in \Re$ (real-valued functions), $\mathbf{f}_1, \mathbf{f}_2, \mathbf{g}_1, \mathbf{g}_2 \in \Re^n$ (vector fields)

$$\begin{cases}
[r_1\mathbf{f}_1 + r_2\mathbf{f}_2, \mathbf{g}_1] = r_1[\mathbf{f}_1,\mathbf{g}_1] + r_2[\mathbf{f}_2,\mathbf{g}_1]\\
[\mathbf{f}_1, r_1\mathbf{g}_1 + r_2\mathbf{g}_2] = r_1[\mathbf{f}_1,\mathbf{g}_1] + r_2[\mathbf{f}_2,\mathbf{g}_1]
\end{cases} \tag{5.9}$$

(2) Skew Commutative

$$[\mathbf{f},\mathbf{g}] = -[\mathbf{g},\mathbf{f}] \tag{5.10}$$

(3) Jacobian Identity : $\mathbf{f},\mathbf{g},\mathbf{h} \in \Re^n$

$$[\mathbf{f},[\mathbf{g},\mathbf{h}]] + [\mathbf{g},[\mathbf{h},\mathbf{f}]] + [\mathbf{h},[\mathbf{f},\mathbf{g}]] = \mathbf{0} \tag{5.11}$$

A.5.6. Properties.

$$\begin{cases}
L_{\mathbf{f}}\lambda(\mathbf{x}) = \langle \nabla\lambda(\mathbf{x}), \mathbf{f}(\mathbf{x})\rangle\\
L_{\mathbf{f}}\omega(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})[\nabla\omega^T]^T + \omega(\mathbf{x}).\nabla\mathbf{f}\\
[\mathbf{f},\mathbf{g}](\mathbf{x}) = \nabla\mathbf{g}.\mathbf{f}(\mathbf{x}) - \nabla\mathbf{f}.\mathbf{g}(\mathbf{x})
\end{cases}
\qquad
\begin{cases}
L_{\mathbf{f}}^{k}\lambda(\mathbf{x}) = L_{\mathbf{f}} L_{\mathbf{f}}^{k-1}\lambda(\mathbf{x})\\
ad_{\mathbf{f}}^{k}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, ad_{\mathbf{f}}^{k-1}\mathbf{g}](\mathbf{x}), \quad k \ge 1\\
ad_{\mathbf{f}}^{0}\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x})
\end{cases}$$

(1) if $\lambda \in \Re$, $\mathbf{f} \in \Re^n$

$$L_{\mathbf{f}}\nabla\lambda(\mathbf{x}) = \nabla L_{\mathbf{f}}\lambda(\mathbf{x}) \tag{5.12}$$

(2) if $\alpha, \lambda \in \Re$, $\mathbf{f},\mathbf{g} \in \Re^n$

$$L_{\alpha\mathbf{f}}\lambda(\mathbf{x}) = (L_{\mathbf{f}}\lambda).\alpha(\mathbf{x}) \tag{5.13}$$

(3) if $\lambda \in \Re$, $\mathbf{f},\mathbf{g} \in \Re^n$

$$\begin{cases}
L_{[\mathbf{f},\mathbf{g}]}\lambda(\mathbf{x}) = L_{\mathbf{f}} L_{\mathbf{g}}\lambda - L_{\mathbf{g}} L_{\mathbf{f}}\lambda\\
\langle \nabla\lambda, [\mathbf{f},\mathbf{g}]\rangle = L_{\mathbf{f}}\langle \nabla\lambda, \mathbf{g}\rangle - L_{\mathbf{g}}\langle \nabla\lambda, \mathbf{f}\rangle
\end{cases} \tag{5.14}$$

(4) if **f, g** : vector fields; $\boldsymbol{\omega}$; co-vector field

$$\langle \boldsymbol{\omega}, [\mathbf{f}, \mathbf{g}] \rangle = \langle L_{\mathbf{f}}\boldsymbol{\omega}, \mathbf{g} \rangle - L_{\mathbf{f}} \langle \boldsymbol{\omega}, \mathbf{g} \rangle \tag{5.15}$$

(5) if $\alpha, \beta \in \Re$, $\mathbf{f}, \mathbf{g} \in \Re^n$

$$[\alpha.\mathbf{f}, \beta.\mathbf{g}] = \alpha.\beta.[\mathbf{f}, \mathbf{g}] + (L_{\mathbf{f}}\beta).\alpha.\mathbf{g} - (L_{\mathbf{g}}\alpha).\beta.\mathbf{f} \tag{5.16}$$

(6) if $\alpha, \beta \in \Re$, $\mathbf{f} \in \Re^n$; $\boldsymbol{\omega}$: co-vector field

$$L_{\alpha\mathbf{f}}\beta\boldsymbol{\omega}(\mathbf{x}) = \alpha.\beta.(L_{\mathbf{f}}\boldsymbol{\omega}) + \beta.\langle \boldsymbol{\omega}, \mathbf{f} \rangle.\nabla\alpha + (L_{\mathbf{f}}\beta).\alpha.\boldsymbol{\omega} \tag{5.17}$$

A.5.7. Example.

$$L_{\alpha\mathbf{f}}\lambda(\mathbf{x}) = \frac{\partial \lambda}{\partial \mathbf{x}}.(\alpha.\mathbf{f}) = \left(\frac{\partial \lambda}{\partial \mathbf{x}}.\mathbf{f}\right)\alpha = (L_{\mathbf{f}}\lambda).\alpha$$

$$\begin{aligned}
L_{\alpha\mathbf{f}}\beta\boldsymbol{\omega} &= (\alpha\,\mathbf{f})^T \left[\frac{\partial\,(\beta\boldsymbol{\omega})^T}{\partial \mathbf{x}}\right]^T + \beta\boldsymbol{\omega}\,\frac{\partial(\alpha\mathbf{f})}{\partial \mathbf{x}} = \alpha\,\mathbf{f}^T\left[\beta\frac{\partial\boldsymbol{\omega}^T}{\partial \mathbf{x}} + \frac{\partial\beta}{\partial\mathbf{x}}\boldsymbol{\omega}^T\right]^T + \beta\boldsymbol{\omega}\left(\alpha\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \frac{\partial\alpha}{\partial\mathbf{x}}\mathbf{f}\right) \\
&= \left\{\alpha\,\mathbf{f}^T\left[\beta\frac{\partial\boldsymbol{\omega}^T}{\partial\mathbf{x}}\right]^T + \beta\,\boldsymbol{\omega}.\alpha\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right\} + \alpha\,\mathbf{f}^T\left[\frac{\partial\beta}{\partial\mathbf{x}}\boldsymbol{\omega}^T\right]^T + \beta\,\boldsymbol{\omega}\,\frac{\partial\alpha}{\partial\mathbf{x}}\mathbf{f} \\
&= \alpha\beta\left\{\mathbf{f}^T\left[\frac{\partial\boldsymbol{\omega}^T}{\partial\mathbf{x}}\right]^T + \boldsymbol{\omega}.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right\} + \alpha\,\mathbf{f}^T\left[\frac{\partial\beta}{\partial\mathbf{x}}\boldsymbol{\omega}^T\right]^T + \beta\,\boldsymbol{\omega}\,\frac{\partial\alpha}{\partial\mathbf{x}}\mathbf{f} \\
&= \alpha.\beta.\,L_{\mathbf{f}}\boldsymbol{\omega} + \alpha\mathbf{f}^T\left(\frac{\partial\beta}{\partial\mathbf{x}}\right)^T\boldsymbol{\omega} + \beta\boldsymbol{\omega}\,\frac{\partial\alpha}{\partial\mathbf{x}}\mathbf{f} = \alpha.\beta.\,L_{\mathbf{f}}\boldsymbol{\omega} + \alpha.\,L_{\mathbf{f}}\beta.\boldsymbol{\omega} + \beta\boldsymbol{\omega}\langle d\alpha, \mathbf{f}\rangle
\end{aligned}$$

Appendix

Bibliography

B.1. SLIDING-MODE CONTROL

AMBROSINO G., CELENTANO G. and GAROFALO F., 1984, Variable Structure Model Reference Adaptive Control Systems. *Int. J. Control*, **39/6,** 1339-1349.

BAILEY E. and ARAPOSTATHIS A., 1987, Simple Sliding Mode Control Scheme Applied to Robot Manipulators. *Int. J. Control,* **45/4,** 1197-1209.

BARTOLINI G., and FERRAR A., 1995, On Multi-Input Sliding Mode Control of Uncertain Nonlinear Systems. *Proc. of 34-th Conf. on Decision and Control,* New Orleans, LA, **12-1995**, 2221-2124

BARTOLINI G., FERRAR A. and UTKIN V. I., 1992, Design of Discrete-Time Adaptive Sliding-Mode Control. *Proc. of 31st Conf. on Decision and Control*, Tucson, Arizona, **12-1992**, 2387-2391

BARTOLINI G. and ZOLEZZI T., 1986, Dynamic Output Feedback for Observed Variable-Structure Control Systems. *Syst. Control Lett.,* **7/6,** 189-193.

BARTOLINI G. and ZOLEZZI T., 1985, Variable Structure Nonlinear in the Control Law. *IEEE Trans. Aut. Control,* **AC-30,** 681-684.

BONDAREV A G, BONDAREV S A, KOSTYLEVA N E and UTKIN V I, 1985, Sliding Mode in Systems with Asymptotic State Observers. *Auto. & Remote Control*, **46/6,** 679-684.

BUJA, G. S. and SOULIAEV A., 1988, A Variable Structure Controller. *IEEE Trans. Aut. Control,* **AC-33/2,** 206-209.

CHANG F. J., TWU S. H. and CHANG S., 1990, Adaptive Chattering Alleviation of Variable-Structure Systems Control. *IEE Proc.-D,* **137/1,** 31-39.

CHANG L. W., 1991, A MIMO Sliding Control with a First-order plus Integral Sliding Condition. *Automatica*, **27/5,** 853-858.

CHEN M. S., and TOMIZUKA M., 1989, A New Robust Controller for Linear Systems with Arbitrarily Structured Uncertainties. *Proc. 28-th on Decision and Control, Florida,* **12-1989,** 431-435.

CHEN Y. C., LIN P. L., and CHANG S., 1992, Design of Output Tracking via Variable Structure System for Plants with Redundant Inputs. *IEE Proc.-D,* **139/4,** 421-428.

CHEN Y. C., LIN P. L., and CHANG S., 1992, Output Tracking Design of Affine NonLinear Plant via Variable-Structure System. *IEEE Trans. Auto. Control,* **AC-37/11,** 1823-1828.

CHEN Y. P., and YEUNG K. S., 1991, Sliding Mode Control of Multi-Link Flexible Manipulators. *Int. J. Control,* **54/2,** 257-278.

CHERN T. L., and WU Y. C., 1992, Integral Variable Structure Control Approach for Robot Manipulators. *IEE Proc.-D,* **139/2,** 161-166.

CHERN T. L., and WU Y. C., 1991, Design of Integral Variable Structure Controller and Application to Electrohydraulic Velocity Servosystems. *IEE Proc.-D,* **138/5,** 439-444.

COLEMAN C. P. and GODBOLE D., 1994, A Comparison of Robustness: Fuzzy Logic, PID & Sliding Mode Control. *IEEE*, 1654-1658.

EDWARDS C. and SPURGEON S. K., 1996, Robust Output Tracking using a Sliding-Mode Controller/Observer Scheme. *Int. J. Control*, **64/5**, 967-983.

DAS D., KOTHARI M. L., KOTHARI D. P., and NANDA J., 1991, Variable Structure Control Strategy to Automatic Generation Control of Interconnected Reheat Thermal System. *IEE Proc.-D,* **138/6,** 579-585.

DECARLO R. A., ZAK S. H., and MATTHEWS G. P., 1988, Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial. *Proc. IEEE*, **76,** 212-232

DORLING C. M., and ZINOBER A. S., 1988, Robust Hyperplane Design in MultiVariable Variable Structure Control Systems. *Int. J. Control,* **48/5,** 2043-2054.

DORLING C. M., and ZINOBER A. S., 1986, Two Approaches to Hyperplane Design in MultiVariable Variable-Structure Control Systems. *Int. J. Control,* **44/1,** 65-82.

DODDS S. J. and WALKER A. B., 1991, Sliding-Mode Control System for the 3-Axis Attitude Control of Rigid-Body Spacecraft with unknown Dynamic Parameters. *Int. J. Control*, **54/4,** 737-761.

DRAZENOVIC B., 1969, The Invariance Conditions in Variable Structure Systems. *Automatica,* **5/3,** 287-295.

EDWARDS C. and SPURGEON S. K., 1996, Robust Output Tracking using a Sliding-Mode Controller/Observer Scheme. *Int. J. Control,* **64/5**, 967-983.

EDWARDS C. and SPURGEON S. K., 1995, Sliding Mode Stabilization of Uncertain Systems using only Output Information. *Int. J. Control,* **62/5**, 1129-1144.

EL-GHEZAWI O. M. E., ZINOBER A. S., and BILLING S. A., 1983, Analysis and Design of Variable Structure Systems using a Geometric Approach. *Int. J. Control,* **38/3,** 657-671.

ELMALI H., and OLGAC. N., 1992, Sliding Mode Control with Perturbation Estimation, SMCPE: A New Approach. *Int. J. Control,* **56/4,** 923-941.

ELMALI H., and OLGAC. N., 1992, Robust Output Tracking Control of Nonlinear MIMO Systems via Sliding Mode Technique. *Automatica,* **28/1,** 145-151.

FERNANDEZ B., and HEDRICK J. K., 1987, Control of Multivariable Nonlinear Systems by the Sliding Mode Method. *Int. J. Control,* **46,** 1019-1040.

FU L., 1992, A New Robust MRAC using Variable Structure Design for Relative Degree 2 Plants. *Automatica,* **28/5,** 911-925.

FU L., and LIAO T., 1990, Globally Stable Robust Tracking of NonLinear Systems using Variable-Structure Control and with an Application to a Robotic Manipulator. *IEEE Trans. Aut. Control,* **AC-35/12,** 1345-1350.

FURUTA K., 1991, VSS-Type Self-Tuning Control. **IECON '91**, 2085-2089.

FURUTA K., 1990, Sliding Mode Control of a Discrete System. *System and Control Letters,* **14**, 145-152.

GE W., LIU X., and CHUI Y., 19??, Variable-Structure Predictor Controller with Quasi-Sliding Mode for Systems with Delay, 215-217.

HABIBI S. R., and RICHARDS R. J., 1992, Sliding Mode Control of an Electrically Powered Industrial Robot. *IEE Proc.-D,* **139/2,** 207-225.

HECK B. S., 1991, Sliding-Mode Control for Singularly Perturbed Systems. *Int. J. Control,* **53/4,** 985-1001.

HIKITA H., 1988, Servomechanisms based on Sliding Mode Control. *Int. J. Control,* **48/2,** 435-447.

HONG C., and WU Y. , 1989, Design of Equal Excursion Sliding Regime. *Int. J. Control,* **49/5,** 1615-1631.

HWANG Y.R and TOMIZUKA M, 1994, Fuzzy Smoothing Algorithms for Variable Structure Systems. *IEEE Trans. on Fuzzy Systems*, **2/4**, 277-284.

ISHIGAME A., FURUKAWA T., KAWAMOTO S. and TANIGUCHI T., 1993, Sliding Mode Controller Design Based on Fuzzy Inference for Nonlinear Systems. *IEEE Trans. on Industrial Electronics*, **40/1,** 64-70

ITKIS U., 1976, Control Systems of Variable Structure. *New York, NY: Wiley.*

IZOSIMOV D. B., and BAIDA S. V., 1985, Vector Method of Design of Sliding Motion and Simplex Algorithms. *Aut. Remote Control,* **46/6,** 830-837.

KANTOR J. C., 1989, Non-linear Sliding-Mode Controller and Objective Function for Surge Tanks. *Int. J. Control,* **50/5,** 2025-2047.

KAWAJI S. and MATSUNAGA N., 1991, Fuzzy Control of VSS Type and its Robustness. *IFSA'91 Brussels*, 7-12.

KOTTA U., 1989, Comments on the Stability of Discrete-Time Sliding Mode Control Systems. *IEEE Trans. Aut. Control,* **AC-34,** 1021-1022.

KRISHNASWAMI V. and RIZZONI G., 1995, Vehicle Steering System State Estimation using Sliding Mode Observers. *Proc. 34-th Conf. on Decision & Control, New Orleans, LA,* **Dec. 1995**, 3391-3396.

LEE D. S., KIM M. G., KIM H. K., and YOUN M. J., 1991, Controller Design of Multivariable Variable Structure Systems with Nonlinear Switching Surfaces. *IEE Proc.-D,* **138/5,** 493-499.

LI Y., CHWEE K., SMITH D. J., GRAY G. and SHARMAN K. C., 1996, Genetic Algorithm Automated Approach to the Design of Sliding-Mode Control System. *Int. J. Control*, **63/4**, 721-739.

LIU J., and YUAN K., 1989, On Tracking Control for Affine NonLinear Systems by Sliding Mode. *Syst. Control Lett.,* **13,** 439-443.

LOZGACHEV G. I., 1972, The Construction of a Lyapunov Function for Variable-Structure Systems. *Aut. Remote Control,* **33/8,** 1391-1393.

MEDANIC J. V. and DIONG B. M., 1992, Dynamic Output Feedback Variable-Structure Control for System Stabilization. *Int. J. Control,* **56/3,** 607-630.

MEDANIC J. V. and DIONG B. M., 1990, Robust Implementation of a Variable-Structure Control Scheme. *Proc. 29th IEEE Conf. Dec. & Ctl.,* **1990**, 3458-3463.

MILOSAVLJEVIC C., 1985, General Conditions for the Existence of a Quasi-Sliding Mode on the Switching Hyperplane in Discrete Variable Structure Systems. *Aut. Remote Control,* **46/3, pt. 1,** 307-314.

NARENDRA K. S., and BOSKOVIC J. D., 1992, A Combined Direct, Indirect, and Variable Structure Method for Robust Adaptive Control. *IEEE Trans. Aut. Control,* **AC-37/2,** 262-268.

NGUYEN D. K., 1994, A New Variable-Structure System Controller Design. *ME Thesis*, University of Technology, Sydney.

NGUYEN H. T. and NGUYEN D. K., 1997, Robust Sliding-Mode Control for Uncertain Linear Systems. *Control '97,* Sydney, Australia, 20-22 Oct.'97, to appear.

NGUYEN H. T. and NGUYEN D. K., 1995, Continuous Sliding-Mode Control. *Control '95*, Melbourne, Australia, 23-25 Nov.'95, 57-60.

PALM R., 1992, Sliding Mode Fuzzy Control. *IEEE Int. Conf. on Fuzzy Systems 1992, Fuzz-IEEE'92 - Proc. San Diego, March 8-12,* ***519-526****.*

PANICKER K. S. M., AHSON S. I., and BHATIA C. M., 1985, The Transient Stabilization of a Synchronous Machine using Variable-Structure Systems Theory. *Int. J. Control,* **42/3,** 715-724.

PARK G., CHOI J., and LEE S., 19??, Design of a Variable-Structure Controller for Drum Motor in Home VCR.

PIEPER . K. and SURGENOR B. W., 1992, Optimal Discrete Sliding Mode Control with Application. *IEEE* 1992, 916-921.

QIJIE Z., and CHUNYI S., 1993, An Adaptive Sliding Mode Control Scheme for Robot Manipulators. *Int. J. Control,* **57/2,** 261-271.

RYAN E. P., 1983, A Variable Structure Approach to Feedback Regulation of Uncertain Dynamical Systems. *Int. J. Control,* **38/6,** 1121-1134.

SARPTURK S. Z., ISTEFANOPULOS Y., and KAYNAK O., 1987, On the Stability of Discrete Time Sliding Mode Control Systems. *IEEE Trans. Aut. Control,* **32,** 930-932.

SINGH S. N., and IYER A., 1988, Variable-Structure Control of Decouplable Systems and Attitude Control of Spacecraft in Presence of Uncertainty. *Proc. Amer. Ctl. Conf.,* **1988,** 2238-2243.

SIRA-RAMIREZ H., 1991, Non-linear Discrete Variable-Structure Systems in Quasi-Sliding Mode. *Int. J. Control,* **54/5,** 1171-1187.

SIRA-RAMIREZ H., 1996, On the Sliding-Mode Control of Multivariable Nonlinear Systems. *Int. J. Control,* **64/4,** 745-765.

SIRA-RAMIREZ H., 1989, Sliding Regimes in General Non-linear Systems: a Relative Degree Approach. *Int. J. Control,* **50/4,** 1487-1506.

SIRA-RAMIREZ H., 1989, Nonlinear Variable Structure in Sliding Mode: The General Case. *IEEE Trans. Aut. Control,* **AC-34/11,** 1186-1188.

SIRA-RAMIREZ H., 1988, Differential Geometric Methods in Variable-Structure Control. *Int. J. Control,* **48/4,** 1359-1390.

SIVARAMAKRISHNAN, A. Y., HARIHARAN M. V., and SRISAILAM M. C., 1984, Design of Variable Structure Load-Frequency Controller using Pole Assignment Technique. *Int. J. Control,* **40/3,** 487-498.

SLOTINE J. J. E., HEDRICK J. K. and MISAWA E. A., 1987, On Sliding Observers for Non-linear Systems. *J. of Dynamic Systems, Measurement, and Control,* **109,** 245-252.

SLOTINE J. J. E., HEDRICK J. K. and MISAWA E. A., 1986, On Sliding Observers for Non-linear Systems. *American Control Conference,* **June 1986,** 1794-1800.

SLOTINE J. J., 1984, Sliding Controller Design for Nonlinear Systems. *Int. J. Control,* **40,** 421-434.

SLOTINE J. J. and SASTRY S. S., 1983, Tracking Control of Non-linear Systems using Sliding Surfaces, with Application to Robot Manipulators. *Int. J. Control,* **38,** 465-492.

SPURGEON S. K., 1992, Hyperplane Design Techniques for Discrete-time Variable-Structure Control Systems. *Int. J. Control,* **55/2,** 445-456.

SPURGEON S. K., 1991, Choice of Discontinuous Control Component for Robust Sliding Mode Performance. *Int. J. Control,* **53/1,** 163-179.

SPURGEON S. K., 1991, On the Development of Discrete-Time Sliding Mode Control Systems *International Conference on COntrol '91,* UK, 505-510.

SPURGEON S. K., 1991, Sliding Mode Control Design for Uncertain Discrete-Time Systems. *Proc. 30-th Conf. Decision & Control, Brighton, England,* **Dec.'91**, 2136-2141.

SPURGEON S. K., 1990, Robust Variable Structure Control of Model Reference Systems. *IEE Proc.-D,* **137/6,** 341-348.

UTKIN V. I., 1992, Sliding Modes in Control and Optimization. *Springer-Verlag.*

UTKIN V. I., and DRAKUNOV S. V., 1992, Sliding Mode Control in Dynamic Systems. *Int. J. Control*, **55/4,** 1029-1037.

UTKIN V. I., and DRAKUNOV S. V., 1989, On Discrete-Time Sliding Mode Control. *Preprint of IFAC COnf. on Nonlinear COntrol, Capri, Italy*.

UTKIN V. I., BONDAREV A. G.,BONDAREV S. A., and KOSTYLEVA N. E., 1985, Sliding Modes in Systems with Assymptotic State Observers. *Aut. Remote Control,* **??/?,** 679-684.

UTKIN V. I., 1983, Variable Structure Systems - Present and Future. *Aut. Remote Control,* **44/9,** 1105-1120.

UTKIN V. I. and LUKYANOV A. G., 1982, Optimal Linear Systems with Degenerate Criterion. *Aut. Remote Control,* **7,** 872-879.

UTKIN V. I., 1978, Sliding Modes and Their Application in Variable Structure Systems. *Moscow, Soviet Union: MIR Publishers.*

UTKIN V. I. and YANG K. D., 1978, Methods for Construction of Discontinuity Planes in MultiDimensional Variable Structure Systems. *Aut. Remote Control,* **39/10,** 1466-1470.

UTKIN V. I., 1977, Survey Paper - Variable Structure Systems with Sliding Modes. *IEEE Trans. Aut. Control,* **AC-22/2,** 212-222.

WHITE B. A., and SILSON P. M., 1984, Reachability in Variable Structure Control Systems. *IEE Proc.-D* **131/3**, 85-91.

WHITE B. A., 1983, Reduced Order Switching Functions in Variable Structure Control Theory. *IEE Proc.-D,* **130,** 33-40.

WON M. and HEDRICK J. K., 1996, Multiple-Surface Sliding Control of a Class of Uncertain Nonlinear Systems. *Int. J. Control,* **64/4**, 693-706.

XIAOHAO X., YAOHUA W., and WENHU H., 1990, Variable Structure Control Approach of Decentralized Model-Reference Adaptive Systems. *IEE Proc.-D,* **137/5,** 302-306.

XU J., HASHIMOTO H. and HARASHIMA F., 1989, On Design of VSS Observer for Non-linear Systems. *Trans. of Society of Instrument and Control Engineers,* **25/2**, 20-26.

YAZ E. and AZEMI A., 1993, Variable-Structure Observers with a Boundary for correlated Noise/Disturbance Models and Disturbance Minimization. *Int. J. Control,* **57/5,** 1191-1206.

YAZ E. and AZEMI A., 1991, Variable-Structure Observers for Nonlinear Models with Unbounded Noise and Measurement Uncertainties. *Proc. 10th American Control Conference,* **1991,** 43-48.

YEUNG K. S., and CHEN Y. P., 1990, Sliding Mode Controller Design of a Single-Link Flexible Manipulator under Gravity. *Int. J. Control,* **54/1,** 101-117.

YEUNG K. S., and CHEN Y. P., 1989, Regulation of a One-Link Flexible Robot Arm using Sliding Mode Technique. *Int. J. Control,* **49/6,** 1065-1978.

YEUNG K. S., and CHEN Y. P., 1988, A New Controller Design for a Manipulators using a Theory of Variable Structure Systems. *IEEE Trans. Aut. Control,* **AC 33/2,** 200-206.

YOUNG K. K. D., and KWANTY H. G., 1982, Variable Structure Servomechanism Design and Applications to OverSpeed Protection Control. *Automatica,* **18/4,** 385-400.

YOUNG K. K. D., 1978, Design of Variable Structure Model Following Systems. *IEEE Trans. Aut. Control,* **AC-22/6,** 1079-1085.

YOUNG K. K. D., 1978, Controller Design for a Manipulator using a Theory of Variable Structure Systems. *IEEE Trans. Syst. Man Cybern.,* **8/2,** 101-109.

YOUNG K. K. D., 1977, Asymptotic Stability of Model Reference Systems with Variable Structure Control. *IEEE Trans. Aut. Control,* **AC-22,** 279-281.

ZHOU F., and FISHER D. G., 1992, Continuous Sliding Mode Control. *Int. J. Control,* **55/2,** 313-327.

B.2. OPTIMIZATION & NUMERICAL METHODS AND MATRIX THEORY

ATHANS M., and JOHNSON T. L., 1970, On the Design of Optimal Constrained Dynamic Compensators for Linear Constant Systems. *IEEE Trans. Aut. Control,* **AC-15/12,** 785-793.

ATHANS M., 1968, The Matrix Minimum Principle. *Inf. Control,* **11,** 592-606.

BINGULAC S. P., CUK N. M. and CALOVIC M. S., 1975, Calculation of Optimum Feedback Gains for Output-Constrained Regulator. *IEEE Trans. Aut. Control,* **AC-20/2,** 164-166.

BINGULAC S. P., CUK N. M. and CALOVIC M. S., 1973, A Sequential Algorithm for Calculation of Optimal Output-Constrained Feedback Regulators. *11th Annual Conf. Ckt & Ctl.,* **1973**, paper III B.4.

BINGULAC S. P., CUK N. M. and CALOVIC M. S., 1973, An Output Feedback Proportional-Plus-Integral Regulator for Automatic Generation Control. *IEEE PES Summer Meeting,* **1973,** paper C.73-489-2.

BROYDEN C. G., 1967, Quasi-Newton Methods and their Application to Function Minimisation. *Math. Comp.,* **22,** 368-381.

CHEN Y. H. and DESROCHERS A. A., 1993, Minimum-Time Control Laws for Robotic Manipulators. *Int. J. Control,* **57/1,** 1-27.

CHOI S. S., and SIRISENA H. R., 1974, Computation of Optimal Output Feedback Gains for Linear Multivariable Systems. *IEEE Trans. Aut. Control,* **AC-19/6,** 257-258

FLETCHER R., 1987, Practical Methods of Optimization, 2ed. *John Wiley & Sons.*

FLETCHER R., 1970, A New Approach to Variable Metric Algorithm. *Computer J.,* **13,** 317-322.

FLETCHER R., and POWELL M. J. D., 1963, A Rapidly Convergent Descent Method for Minimization. *Computer J.,* **6,** 163-168.

GEERING. H. P., 1986, Time-Optimal Motions of Robots in Assembly Tasks. *IEEE Trans. Auto. Control,* **AC-31/6,** 512-518.

GEERING. H. P., 1976, On Calculating Gradient Matrices. *IEEE Trans. Auto. Control,* **AC-21/8,** 615-616.

GILL P. E., MURRAY W, and WRIGHT M H, 1981, Practical Optimization. *Academic Press.*

GOLDFARB D., 1970, A Family of Variable Metric Updates Derived by Variational Means. *Mathematics of Computing,* **24,** 23-26.

HARKARA N., PETKOVSKI D., and GAJIC Z., 1989, Recursive Algorithm for the Optimal Static Output Feedback Control Problem of Linear Weakly-Coupled Systems. *Int. J. Control,* **50/1,** 1-11.

HORISBERGER H. P., and BELANGER P. R., 1974, Solution of the Optimal Constant Output Feedback Problem by Conjugate Gradients. *IEEE Trans. Aut. Control,* **AC-19/8,** 434-435.

HUANG L., and LI Z., 1989, Fundamental Theorem for Optimal Output Feedback Problem with Quadratic Performance Criterion. *Int. J. Control,* **50/6,** 2341-2347.

KUHN U., and SCHIMIDT G., 1987, Fresh Look into the Design and Computation of Optimal Output Feedback Controls for Linear Multivariable Systems*. Int. J. Control,* **46/1,** 75-95.

KWAKERNAAK H., and SIVAN R., 1972, Linear Optimal Control Systems. *New York* : *Wiley-Interscience***,** 108-110.

LANCASTER P. and TISMENETSKY M., 1985, The Theory of Matrices, 2-nd Ed. *Academic Press*.

LEVINE W. S., JOHNSON T. L., and M. ATHANS, 1971, Optimal Limited State Variable Feedback Controllers for Linear Systems. *IEEE Trans. Aut. Control,* **AC-16/12,** 785-793.

LEVINE W. S. and M. ATHANS, 1970, On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems. *IEEE Trans. Aut. Control,* **AC-15/2,** 44-48.

MENDEL J. M., 1974, A Concise Derivation of Optimal Constant Limited State Feedback Gains. *IEEE Trans. Aut. Control,* **AC-19/8,** 447-448.

POWELL M. J. D., 1962, An Iterative Method for Finding Stationary Values of a Function of Several Variables. *Comp. J.,* **5,** 147-151.

RAO S. S., 1979, Optimization : Theory and Applications. *Wiley Eastern Ltd.*

ROSENBROCK H. H., 1960, An Automatic Method for Finding the Greatest or Least Value of a Function. *Comp. J.,* **3**, 175-184.

SHANNON D. F., 1970, Condition of Quasi-Newton Methods for Function Minimization. *Math. Comp.,* **24,** 647-656.

TOIVONEN H. T., 1985, A Globally Convergent Algorithm for the Optimal Constant Output Feedback problem. *Int. J. Control,* **41/6,** 1589-1599.

B.3. FUZZY IDENTIFICATION AND CONTROL

BRAAE M. and RUTHERFORD D. A., 1979, Theoretical and Linguistic Aspect of the Fuzzy Logic Controller. *Automatica*, **15**, 553-577.

BRAAE M. and RUTHERFORD D. A., 1978, Fuzzy Relations in a Control Setting. *Kibernetes,* **7/3**, 185-188.

CHEN C.L., CHEN P. C. and CHEN C. K., 1993, Analysis and Design of Fuzzy Control System. *Fuzzy Sets & Systems*, **57**, 125-140.

DALEY S. and GILL K. F., 1986, A Design Study of a Self-Organizing Fuzzy Logic Controller. *Proc. Inst. Mech. Eng,* **200/1,** 59-69.

FARINWATA S. S., 1994, On the Stability of Fuzzy Control Rulebase for a Nonlinear Process. *IEEE* 924-929.

HWANG G. C. and LIN S. C., 1992, A stability Approach to Fuzzy Control Design for Nonlinear Systems. *Fuzzy Sets & Systems*, **48**, 279-287.

KAWAJI S., MAEDA T., and MATSUNAGA N., 1991, Fuzzy Control using Knowledge Acquired from PD Control. *IECON '91*, 1549-1554.

KING P. J. and MANDANI E. H., 1977, The Application of Fuzzy Control Systems to Industrial Processes. *Automatica*, **13,** 235-242.

LEE C. C., 1990, Fuzzy Logic in Control Systems : Fuzzy Logic Controller. *IEEE Trans. on Syst. Man, Cybern.,* **20/2,** 404-435.

MAMDANI E. H., OSTERGAARD J. J., and LEMBESSIS E., 1984, Use of Fuzzy Logic for Implementing Rule-Based Controllers for Industrial Processes. *Fuzzy Sets and Decision Analysis,* **20,** Zimmermann H. J., Zadeh L. A. and Gaines B. R. (eds.), N. Holland, Amsterdam.

MAMDANI E. H., 1983, Process Control using Fuzzy Logic. *Designing for Human-Computer Communication*, **Academic Press**, London.

MAMDANI E. H., 1977, Application of Fuzzy Logic to Approximate Reasoning Using Linguistic Systhesis. *IEEE Trans. Comput.*, **26/12**, 1182-1191.

MAMDANI E.H., and ASSILIAN S., 1975, An Experiment in Linguistic Systhesis with a Fuzzy Logic Controller. *Int. J. of Man-Machine Studies,* **7,** 1-13.

MAMDANI E. H., 1974, Applications of Fuzzy Algorithms for Control of Simple Dynamic Plant. *Proc. of Institution of EE,* **121,** 1585-1588.

MIZUMOTO M., 1992, Realization of PID Controls by Fuzzy Control Methods. *IEEE* **1992**, 709-715.

PALM R, REINFRANK M. and STORJOHANN K, Towards Second Generation Applications of Fuzzy Control. **549-552**.

PROCYK T. J. and MAMDANI E. H., 1979, A Linguistic Self-Organizing Process Controller. *Automatica,* **15**, 15-30.

SUGENO M. and TANAKA K., 1991, Successive Identification of a Fuzzy Model and its Applications to Prediction of a Complex System. *Fuzzy Sets & Systems*, **42**, 315-334.

TANAKA K and SUGENO M., 1992, Stability Analysis and Design of Fuzzy Control Systems. *Fuzzy Sets and Systems*, **45**, 135-156.

TAKAGI T. and SUGENO M, 1985, Fuzzy Identification of Systems and its Application to Modelling and Control. *IEEE Trans. on Syst., Man and Cyb,* **15/1,** 116-132.

TONG R. M., 1978, Analysis and Control of Fuzzy Systems using Finite Discrete Relations. *Int. J. Control*, **27**, 431-440.

TONG R. M., 1977. A Control Engineering Review of Fuzzy Systems. *Automatica*, **13**, 559-569.

YAMASHITA T., KATOH R., SINGH and HORI T., 1991, Stability Analysis of Fuzzy Control System Applying Conventional Methods. *IEEE*, 1579-1584.

ZADEH L. A., 1973, Outline of a New Approach to the Analysis of Complex Systems and Decision Processes. *IEEE Trans. on Syst., Man and Cyern.,* **3/1,** 28-44.

ZADEH L. A., 1968, Fuzzy Algorithms. *Information and Control,* **12,** 94-102.

ZADEH L. A., 1965, Fuzzy Sets. *Information and Control,* **8,** 338-353.

B.4. ROBUST CONTROL AND OPTIMAL H_∞ CONTROL

ACKERMANN J. E., 1984, Robustness against Sensor Failures. *Automatica,* **20/2,** 211-215.

BARMISH B. R., CORLESS M. and LEITMANN G., 1983, A New Class of Stabilizing Controllers for Uncertain Dynamical Systems. *SIAM J. Ctl. & Opti.,* **21/2,** 246-255.

BARMISH B. R., and LEITMANN G., 1982, On Ultimate Boundedness Control of Uncertain Systems in the Absence of Matching Assumptions. *IEEE Trans. Aut. Control,* **AC-27/1,** 153-158.

BEHTASH S., 1990, Robust Output Tracking for NonLinear Systems. *Int. J. Control,* **51/6,** 1381-1407.

CHEN Y. H., LEITMANN G. and KAI X. Z., 1991, Robust Control Design for InterConnected Systems with Time Varying Uncertainties. *Int. J. Control,* **54/5,** 1119-1142.

CHEN Y. H., 1990, State Estimation for NonLinear Uncertain Systems: A Design based on properties related to the Uncertainty Bound. *Int. J. Control,* **52/5,** 1131-1146.

CHEN Y. H., and LEITMANN G., 1987, Robustness of Uncertain Systems in the Absence of Matching Assumptions. *Int. J. Control,* **45/5,** 1527-1542.

CHEN Y. H., 1986, On the Deterministic Performance of Uncertain Dynamical Systems. *Int. J. Control,* **43/5,** 1557-1579.

CHEN Y. H., 1985, Deterministic Control for A New Class of Uncertain Dynamical Systems. *Proc. 24th Conf. Dec. & Ctl.,* **12-1985,** 1302-1306.

CORLESS M. J., and LEITMANN G., 1981, Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems. *IEEE Trans. Aut. Control,* **AC-26/5,** 1139-1144.

DOYLE J. C., FRANCIS B. A. and TANNENBAUM A. R., 1992, Feedback Control Theory. *Maxwell Macmillan*.

DOYLE J. C., GLOVER K., KHARGONEKAR P. P., and FRANCIS B. A., 1989, State-Space Solutions to Standard H_2 and H_∞ Control problems. *IEEE Trans. on Auto. Control*, **34/8**, 831-847.

Francis B. A., 1988, A Course in H_∞ Control Theory. *Springer-Verlag*.

GILBERT E. M., and HA I. J., 1987, Robust Tracking in NonLinear Systems. *IEEE Trans. Aut. Control,* **AC-32/9,** 763-771.

GLOVER K and DOYLE J. C., 1988, State-Space Formula for All Stabilizing Controllers that Satisfy an H_∞-norm Bound and Relations to Risk Sensitivity. *Sys.* & *Control Let.*, **11**, 167-172.

GUTMAN S., 1979, Uncertain Dynamical Systems - A Lyapunov Min-Max Approach. *IEEE Trans. Aut. Control,* **AC-24/3,** 437-443.

HECK B. S., and HADDAD A. H., 1989, Singular Perturbation in Piecewise-Linear Systems. *IEEE Trans. Aut. Control,* **AC-34/1,** 87-90.

HECK B. S., and HADDAD A. H., 1988, Singular Perturbation Analysis of Linear Systems with Scalar Quantized Control. *Automatica*, **24/6,** 755-764.

KOKOTOVIC P. V., SAKSENA V. R., and REILLY J. O., 1984, Singular Perturbations and Time-Scale Methods in Control Theory : Survey 1976-1983. *Automatica,* **20/3,** 273-293.

LIN F., BRANDT R. D., and SUN J., 1992, Robust Control of NonLinear Systems: Compensating for Uncertainty. *Int. J. Control,* **56/5,** 1453-1459.

KRAVARIS C., and PALANKI S., 1988, A Lyapunov Approach for Robust NonLinear State Feedback Synthesis. *IEEE Trans. Aut. Control,* **AC-33/12,** 1188-1191.

KUR H., OHKA N., OHBA M., HOSOE S., and ZHANG F., 1989, Application of H∞ Optimal Design to Automotive Fuel Control. *June 1989 Amer. Cont. Conf.,* **3,** 1957-1962.

KURAOKA H, OHKA N, OHBA M., HOSOE S. and ZHANG F, 1989, Application of H_∞ Optimal Design to Automative Fuel Control, *American Control Conference*, **3**, 1957-1962.

KWAKERNAAK H., 1993, Robust Control and H_∞-Optimization - Tutorial Paper. *Automatica***, 29/2,** 255-273.

MEDANIC J. V., VEILLETTE R. J., and PERKINS W. R., 1989, Robust Stabilization and Disturbance Rejection for Systems with Structured Uncertainty. *Proc. 28^th^ IEEE Conf. Dec. & Ctl.,* **1989,** 936-941.

PETERSEN I. R., 1987, Disturbance Attenuation and H_∞ Optimization: A Design Method Based on the Algebraic Riccati Equation. *IEEE Trans. on Auto. Control,* **32/5**, 427-429.

RYAN E. P., and CORLESS M., 1984, Ultimate Boundedness and Asymptotic Stability of a Class of Uncertain Dynamical System via Continuous and Discontinuous Feedback Control. *IMA J. of Mathematical Control & Information,* **1,** 223-242.

TAKANO M., KUROTANI K. and TAKEDA K., 1991, Application of H_∞ Control to Motor Speed Control System. *IECON '91*, 839-842.

ZHOU K. and KHARGONEKAR P. P., 1988, An Algebraic Riccati Equation Approach to H_∞ Optimization. *Sys.* & *Control Let,* **11,** 85-91.

B.5. ADAPTIVE CONTROL

ASTROM K. J., and WITTENMARK B., 1989, Adaptive Control. *Addison-Wesley*.

CHAN Y. T., 1973, Perfect Model Following with a Real Model. *Proc. Joint Auto. Ctl. Conf.,* **1973,** 287-293.

GOODWIN G. C. and SIN K. S., 1984, Adaptive Filtering Prediction and Control. *Prentice-Hall*.

ERZBERGER H., 1968, Analysis and design of model following systems by state-space techniques. *Proc. Joint Auto. Cont. Conf.,* **1968,** 572-581.

HANG C. C., and TANG H. S., 1981, Model Reference Adaptive Control Using Only Input and Output Measurements. *Int. J. Control,* **34/5,** 853-868.

B.6. NONLINEAR CONTROL

DESCUSSE J., and MOOG C. H., 1987, Dynamic Decoupling for Right-Invertible NonLinear Systems. *Syst. Control Lett,* **8,** 345-349.

FLIESS M, 1990, Generalized Controller Canonical Forms for Linear and NonLinear Dynamics. *IEEE Trans. Aut. Control,* **AC-35/9,** 994-1001.

FLIESS M., and HAZEWINKEL M., Eds., 1986, Algebraic and Geometric Methods in NonLinear Control Theory. *D. Reidel.*

HALL S. A., 1989, Real-Time Control of a NonLinear ElectroMagnetic Actuator. *Automatica,* **25/1,** 17-31.

HERMANN R., and KRENER A. J., 1977, NonLinear Controllability and Observability. *IEEE Trans. Aut. Control,* **AC-22/5,** 728-740.

HUNT L. R., LUKSIC M., and SU R., 1986, Exact Linearizations of Input-Output Systems. *Int. J. Control,* **43/1,** 247-255.

HUNT L. R., SU R., and MEYER G., 1983, Global Transformations of NonLinear Systems. *IEEE Trans. Aut. Control,* **AC-28/1,** 24-31.

ISIDORI A. and BYRNES C. I., 1990, Output Regulation of NonLinear Systems. *IEEE Trans. Aut. Control,* **AC-35/2,** 131-140.

ISIDORI A., 1989, NonLinear Control Systems. *Springer-Verlag.*

ISIDORI A., and BYRNES I., 1988, Local Stabilization of Minimum-Phase NonLinear Systems. *Syst. Control Lett.*, **11,** 9-17.

ISIDORI A., and BYRNES I., 1983, Linearization by Output Injection and NonLinear Observers. *Syst. Control Lett.*, **3/6,** 47-52.

KOBAYASHI H., and SHIMEMURA E., 1989, Set-Point Changing for NonLinear Systems. *Int. J. Control,* **50/6,** 2397-2406.

KOKOTOVIC P. V., TAYLOR D. G., and KANELLAKOPOULOS I., 1988, Adaptive Regulation of NonLinear Systems with Unmodelled Dynamics. *Proc. Amer. Ctl. Conf.*, **1988,** 360-365.

NIJMEIJER H., 1986, Right-Invertibility for a Class of NonLinear Control System : A Geometric Approach. *Syst. Control Lett.*, **7/4,** 125-132.

RUGH W. J., 1986, An Extended Linearization Approach to NonLinear System Inversion. *IEEE Trans. Aut. Control,* **AC-31/8,** 725-733.

SASTRY S. S., and ISIDORI A., 1989, Adaptive Control of Linearizable Systems. *IEEE Trans. Aut. Control,* **AC-34/11,** 1123-1131.

TSENG H. C., and KOKOTOVIC P. V., 1988, Tracking and Disturbance Rejection in NonLinear Systems: the Integral Manifold Approach. *Proc. Conf. Dec. & Ctl.*, **1988,** 459-463.

WALCOTT B. L. and ZAK S. H., 1987, State Observation of Nonlinear Uncertain Dynamical Systems. *IEEE Trans. on Auto. Control,* **AC-32/2,** 166-170.

WALCOTT B. L., CORLESS M. J. and ZAK S. H., 1987, Comparative study of Non-Linear State-Observation Techniques. *Int. J.. Control,* **45/6,** 2109-2132.

WALCOTT B. L. and ZAK S. H., 1986, Observation of Dynamical Systems in the Presence of Bounded Nonlinearities/Uncertainties. *Proceeding of 25th Conference on Decision and Control, Athens, Greece,* **December 1986,** 961-966.

ZAK H. S., and MACCARLEY C. A., 1986, State-Feedback Control of NonLinear Systems. *Int. J. Control,* **43/5,** 1497-1514.

B.7. DIGITAL SIGNAL PROCESSING

BATEMAN A. and YATES W., 1988, Digital Signal Processing. *Pitman,* London.

BRIGHAM E. O., 1988, The Fast Fourrier Transform and its Applications. *Prentice Hall,* New Jersey.

BURRUS C. S. and PARKS T. W., 1985, DFT/FFT and Convolution Algorithms. *John Wiley & Sons,* New York

PARKS T. W. and BURRUS C. S., 1987, Digital Filter Design. *John Wiley & Sons,* New York.

LOCKHART G. B. and CHEETHAM B. M. G., 1989, Basic Digital Signal Processing. *Butterworths*, London.

ANTONIOU A., 1993, Digital Filters. *Mc Graw-Hill*, New York.

IFEACHOR E. C. and JERVIS B. W., 1993, Digital Signal Processing: A Practical Approach. *Addison-Wesley*, Workingham, England.

PROAKIS J. G. and MANOLAKIS D. G., 1996, Digital Signal Processing: Principles, Algorithms and Applications, 3-rd Edition. *Mac-Millan*, New York.

OPPENHEIMER A. V. and SCHAFER R. W., 1989, Discrete-Time Signal Processing. *Prentice Hall*, New Jersey.

B.8. NEURAL NETWORK

CHARALAMBOUS C, 1992, Conjugate Gradient Algorithm for Efficient Training of Artificial Neural Networks. *IEE Proceeding-G,* **139,** 301-310.

CUN Y L, 1988, A Theoretical Framework for Back-Propagation. *Technical Report M5S 1A4, University of Toronto,* 21-28.

JOHMHANSSON E M, DOWLA F U, and GOODMAN D M, BackPropagation Learning for MultiLayer Feed-Forward Neural Networks Using the Conjugate Gradient Method. *International Journal of Neural Systems*, **2**, 291-301.

LAPEDES, A., AND FARBER, R., 1988, How Neural Nets Work. *Neural Information Processing Systems,* edited by D. Z. Anderson (New York : American Institute of Physics) 442-456.

LIPPMANN, R. P., 1987, An Introduction to computing with Neural Nets. *IEEE Acoustics, Speech and Signal Processing Magazine,* April, 4-22.

MOLLER M F, 1993, A Scaled Conjugate Gradient Algorithm for Fast Supervised Learning. *Neural Networks,* **6**, 525-533.

NGUYEN D. H. and WIDROW B., 1991, Neural Networks for Self-Learning Control Systems. *Int. J. Control*, **54/6**, 1439-1451.

RUMELHART D E, HINTON GE, and WILLIAMS R J, 1986, Learning Internal Representations by Error Propagation. *Parallel Distributed Processing,* **1,** 318-362

WINSTON, P. H., 1992, Artificial intelligence 3ed., **Chapter 22**. *Addison-Wesley.*

B.9. GENERAL

ACKERMANN J. E., 1980, Parameter Space Design of Robust Control Systems. *IEEE Trans. Aut. Control,* **AC-25/6,** 1058-1072.

ACKERMANN J. E., 1977, On the Synthesis of Linear Control Systems with Specified Characteristics. *Automatica*, **13/1,** 89-94.

BROCKETT R., MILLMAN S., and SUSSMANN H. eds., 1983, Differential Geometric Control Theory. *Birkhauser*.

CHAPELLAT H., and BHATTACHARYYA S. P., 1989, An Alternative Proof of Kharitonov's Theorem. *IEEE Trans. Aut. Control,* **AC-34/4,** 448-450.

ITKIS U., 1983, Dynamic Switching of Type-I-II Structure in Tracking Servosystems. *IEEE Trans. Aut. Control,* **AC-28/4,** 531-534.

MASSEY J. L., and SAIN M. K., 1969, Invertibility of Linear Time-Invariant Dynamical Systems. *IEEE Trans. Aut. Control,* **AC-14/2,** 141-149.

MINNICHELLI R. J., ANAGNOST J. J., and DESOER C. A., 1989, An Elementary Proof of Kharitonov's Stability Theorem with Extensions. *IEEE Trans. Aut. Control,* **AC-34,** 995-998.

OGATA K., 1987, Discrete-Time Control Systems. *Prentice-Hall International, Inc.*

SILVERMAN L. M., 1969, Inversion of Multivariable Linear Systems. *IEEE Trans. Aut. Control,* **AC-14/3,** 270-276.

VEGTE J. V., 1994, Feedback Control Systems, 3rd Ed. *Prentice Hall*, New Jersey.

YOUCEF-TOUMI K. and WU S., 1992, Input/Output Linearization using Time Delay Control. *J. Dynamic Syst. Meas. & Control,* **114/3,** 1--19.

YOUCEF-TOUMI K. and BOBBETT J., 1991, Stability of Uncertain Linear Systems with Time Delay. *J. Dynamic Syst. Meas. & Control,* **113/12,** 558-567.

YOUCEF-TOUMI K., and ITO O., 1990, A Time Delay Controller for Systems with Unknown Dynamics. *J. Dynamic Syst. Meas. & Control,* **112/3,** 133-142.

YOUNG K. K. D., KOKOTOVIC P. V., and UTKIN V. I., 1977, A Singular Perturbation Analysis of High-Gain Feedback Systems. *IEEE Trans. Aut. Control,* **AC-22/6,** 931-938.

ZOHDY M. A., and APLEVICH J. D., 1976, A Class of Output Feedback Controllers and Min-Max Design. *IEEE Trans. Aut. Control,* **AC-21/8,** 614-615.

Bibliography