Basic Sliding Mode Controller Design

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1. Introduction

Sliding mode control (SMC) is able to deal with uncertainty and nonlinearity.

In the sliding-mode control theory, control dynamics have 2 sequential modes, the first is the *reaching mode* and the second is the *sliding mode* (Utkin 1977, Utkin 1992). In particular, the Lyapunov sliding condition forces system states to *reach* a hyperplane and keeps them *sliding* on this hyperplane. Essentially, a SMC design is composed of 2 phases: hyperplane design and controller design. A hyperplane is first designed via the pole-placement approach as in the state-space control (Utkin *et al.* 1979), a controller design is then based on the sliding condition. The stability is guaranteed by the sliding condition (Lyapunov Stability Criterion Theorem) *and* by a *stable* hyperplane (stable designer-chosen pole-placement). In the reaching mode, the control dynamics depend on system parameters; but in the sliding mode they depend on the hyperplane, this is the *invariance* property of the sliding mode (Drazenovic 1969).

This paper presents basic sliding mode control theory, so I will use the 2^{nd}-ordered SISO canonical. More general case MIMO higher order can be found in Ref[1].

2. Hyperplane Design

Theorem 1: Hyperplane Design for Canonical Nonlinear Systems

For a n-ordered canonical *linear or nonlinear* system, if the hyperplane-eigenvalue is

$$\lambda_H = \lambda_1 \tag{1}$$

then a hyperplane can be found by

$$s = \mathbf{H}.\mathbf{x}, \quad \mathbf{H} = [h_1, \quad 1] \tag{2}$$

where h is the coefficient of the following polynomial

$$(\lambda - \lambda_1) = \lambda + h_1 \tag{3}$$

Proof

Consider the following n-ordered canonical linear or nonlinear system with output $y = x_1$

$$\begin{cases} \dot{y} = \dot{x}_1 = x_2 \\ \ddot{y} = \dot{x}_2 = \phi(\mathbf{x}), \quad \phi(\mathbf{x}) : \text{linear or nonlinear function of the system state variable } \mathbf{x} \end{cases}$$

and a hyperplane

$$s = \mathbf{Hx} = [h_1, \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h_1 x_1 + x_2$$

In the sliding mode, I have the following linear differential equation

$$s = 0 \;\Rightarrow\; h_1 x_1 + x_2 = 0 \;\Rightarrow\; h_1 y + \dot{y} = 0 \tag{4}$$

then the corresponding characteristic equation is

$$\lambda + h_1 = 0 \tag{5}$$

thus if the roots of this equation Eq.(5) are Hurwitz, then the output decays according to the above linear differential equation Eq.(4).

Q.E.D

3. Sliding Condition

Choose a positive definite function for a Lyapunov candidate (Utkin 1977)

$$V = \tfrac{1}{2}s^2 \Rightarrow \dot{V} = s.\dot{s} \tag{6}$$

Define a *sliding condition* as

$$\boxed{s.\dot{s} < 0} \tag{7}$$

then

$$\left.\begin{array}{l} V > 0 \\ \dot{V} < 0 \end{array}\right\} \Rightarrow V \xrightarrow[t\to\infty]{} 0 \Rightarrow s \xrightarrow[t\to\infty]{} 0 \Rightarrow V \text{ must reduce to } \textit{zero} \Rightarrow s \text{ must reduce toward } \textit{zero}$$

For $s \to 0$ in a finite time to achieve the *sliding mode* $(s = 0)$, the sliding condition should be strictly $s.\dot{s} < -\delta$ where $\delta > 0$. That is the larger δ is, the faster the sliding mode is attained. Strictly speaking, the sliding mode exists only asymptotically.

Hence the sliding condition is analogous to the *Lyapunov's direct stability criterion*.

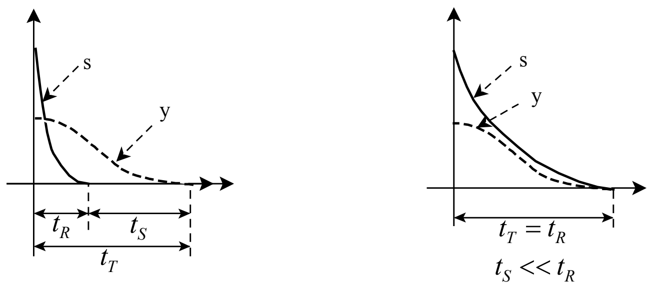

Fig.1.1: Reaching mode and sliding mode

where

s is sliding variable from Eq.(2)

y is system output

$t_T = t_R + t_S$ with

- t_T is total time
- t_R is reaching time $(s.\dot{s} < 0)$
- t_S is sliding time $(s = 0)$

In the figure above, there is a sliding mode in the first case, but practically not in the second.

When $s \neq 0$, the system is in the $\underline{\textit{reaching mode } (s.\dot{s} < 0)}$, and the sliding condition then guarantees the $\underline{\textit{sliding mode } (s = 0)}$.

Remark 1: Negativeness of Sliding Condition

The sliding condition must be *negative enough* to guarantee that the reaching mode terminates in a finite time for the sliding mode to exist

Remark 2: Stability of the Sliding Mode

A stable sliding mode implies a stable system only if the sliding condition is satisfied. The sliding mode stability is based on 6 theorems in Utkin 1977 where the proofs were referred to other works in the Russian literature. In the latest book by Utkin 1992, these 6 theorems were omitted and the sliding-mode stability was based instead on 2 complex theorems. Alternatively, I will propose a simple stability criterion in Section 3.2.

Remark 3: Dimension of Sliding Hyperplane

In the sliding mode, the system effectively becomes another system of reduced order. Therefore, a 2-nd order system has a sliding *line*, a 3-rd order system has a sliding *plane*, a higher order system has a sliding *hyperplane*.

4. Sliding Dynamics

From Eq.(1.1), if the output y and the state vector $\mathbf{x}$ are defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

then I have the following sliding dynamics equation

$$s = 0 \Rightarrow h.x_1 + x_2 = 0 \Rightarrow \dot{y} + h.y = 0 \tag{8}$$

$$\Rightarrow y(t) = y(0).e^{-h.t} \tag{9}$$

Therefore, once $s = 0$, the output state y decays as described by Eq.(9. Note that Eq.(8 is a *characteristic equation* whose root is $-h$, so the hyperplane-eigenvalue is defined as $\lambda_H = [-h]$. This hyperplane-eigenvalue determines the sliding dynamics. The sliding dynamics depend only on this hyperplane-eigenvalue and not on the system parameters or on how the sliding mode is reached: that is the property of invariance in the sliding mode (Drazenovic 1969).

5. Equivalent / Reaching Control

Theorem 2 Continuous SMC Design for Nonlinear Systems

Consider a canonical nonlinear SISO system

$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t).u$

then a continuous SMC control function is determined by

$$\underline{\underline{u = u_e + u_r}} \tag{10}$$

where

- equivalent control
$$u_e = -(\mathbf{Hg})^{-1}\mathbf{Hf}, \tag{10.a}$$
- reaching control
$$u_r = -(\mathbf{Hg})^{-1}.\delta.s \tag{10.b}$$

where

$\mathbf{x}, \mathbf{f}, \mathbf{g} \in \Re^{n\times 1}$, $\mathbf{H} \in \Re^{1\times n}$, $u, s \in \Re$, $\delta \in \Re_{(+)}$: sliding margin.

Proof

Since the system is in the canonical form, direct eigenvalue allocation is applied for a hyperplane $s = \mathbf{Hx} \Rightarrow \dot{s} = \mathbf{H}\dot{\mathbf{x}} = \mathbf{H}(\mathbf{f}+\mathbf{g}.u) = \mathbf{Hg}.\left[u+(\mathbf{Hg})^{-1}\mathbf{Hf}\right]$.

By the above continuous SMC control function, I have

$s\dot{s} = -\delta.s^2 < 0$: the sliding condition is satisfied.

Q.E.D.

6. SMC Stability Criterion for Nonlinear Systems

Based on Theorem 6.1 above, I have the following corollary on a stability test for both linear and nonlinear systems.

Corollary 1: Stability Criterion

For an n-ordered canonical *linear or nonlinear* system, if there exists a control function to satisfy the sliding condition $s.\dot{s} \le 0$ n the hyperplane determined by Theorem 2 then the system is stable.

Proof

The control function satisfying the sliding condition will drive the system into the sliding mode. Then, by Theorem 2 the system is stable.

Q.E.D.

7. Numerical Examples

Consider a system in Zhou *et al.* 1992 which is modified with a reference output:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1x_2 + x_1^2 + \sin(tx_1) + \left(1+\sqrt{|x_1|}\right)u \end{bmatrix} \\ y_{ref} = \ln(t+1) \Rightarrow z = y - y_{ref} = x_1 - \ln(t+1) \end{cases}$$

then

$$\begin{cases} z = x_1 - \ln(t+1) \\ \dot{z} = x_2 - \dfrac{1}{t+1} \\ \ddot{z} = 2x_1x_2 + x_1^2 + \sin(tx_1) + \dfrac{1}{(t+1)^2} + \left(1+\sqrt{|x_1|}\right)u \end{cases}.$$

Choose a hyperplane-eigenvalue

$$\lambda_H = [-2] \Rightarrow \mathbf{H} = [2 \quad 1]$$

then the hyperplane

$$s = \mathbf{H}.\mathbf{z}$$

By Theorem 2, a continuous SMC is determined by

$$u = -\left(1+\sqrt{|x_1|}\right)^{-1}\left\{2\left(x_2 - \frac{1}{t+1}\right) + 2x_1x_2 + x_1^2 + \sin(tx_1) + \frac{1}{(t+1)^2} + \delta.s\right\}.$$

Choose $\delta = 4$, I have

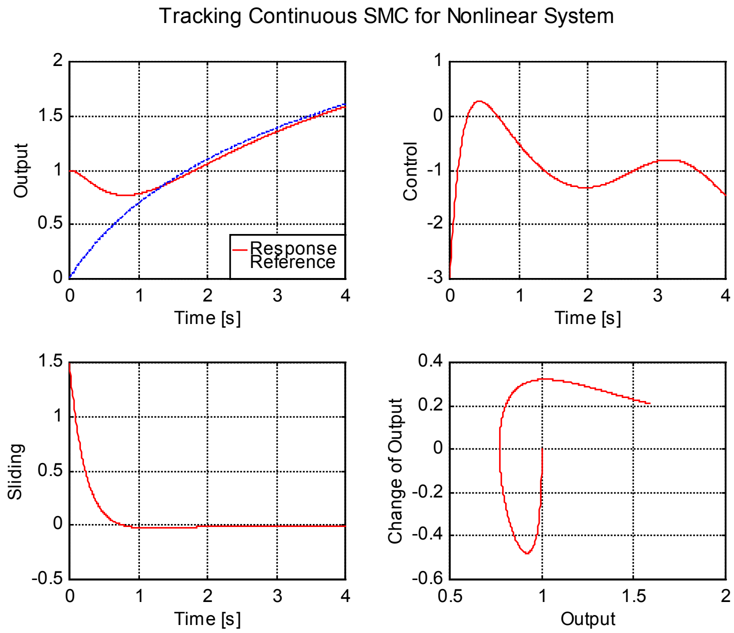

Fig. 6.2: Tracking Continuous Nonlinear SMC for Example 6.3.

References

[1] Duy-Ky Nguyen, *Sliding-Mode Control : Advanced Design Techniques*, PhD Thesis, University of Technology, Sydney, Australia, 1998.