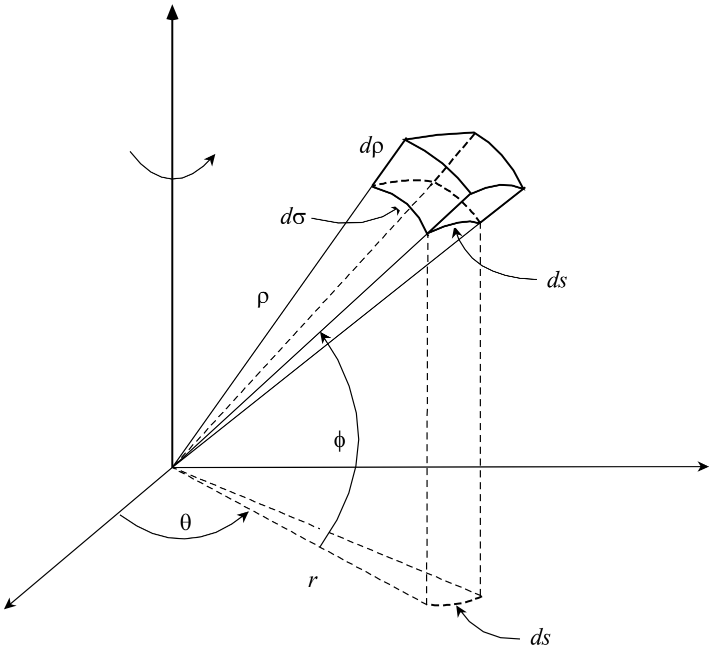
Control Lab Projects: Modelling

Duy-Ky Nguyen 20-03-95



$$r = \rho\cos\phi, \quad d\sigma = \rho\, d\phi, \quad ds = r\, d\theta = \rho\cos\phi\, d\theta, \quad \rho \in [0, R], \quad \phi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad \theta \in [0, 2\pi]$$

$$dV = d\rho . d\sigma . ds = \rho^2 d\rho . \cos\phi\, d\phi . d\theta \;\Rightarrow\; V = \frac{4\pi}{3} R^3$$

$$dI = dV . r^2 = d\theta . \rho^4\, d\rho . \cos^3\phi\, d\phi = d\theta . \rho^4\, d\rho . \cos\phi\left(1 - \sin^2\phi\right) d\phi$$

$$\int_{-\pi/2}^{\pi/2} \cos\phi\left(1 - \sin^2\phi\right) d\phi = \int_{-1}^{1} \left(1 - u^2\right) du = \frac{4}{3}$$

$$I = \frac{4}{3}\pi R^3 \frac{2}{5} R^2 = \frac{2}{5} M R^2$$

Ball and Beam

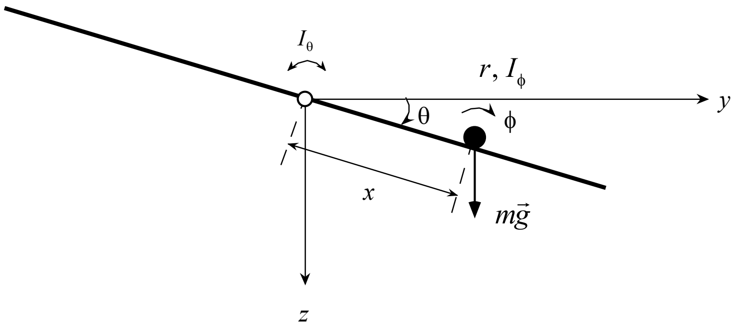
Duy-Ky Nguyen **10-04-1994**

The total co-kinetic energy is

$$T = \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} I_\phi \dot{\phi}^2 + \frac{1}{2} m\left(\dot{y}^2 + \dot{z}^2\right) = \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} m\left(\dot{y}^2 + \dot{z}^2 + \frac{2}{5} r^2 \dot{\phi}^2\right) \quad (1.1)$$

where

$$\left.\begin{aligned} y = x\cos\theta &\Rightarrow \dot{y} = \dot{x}\cos\theta - x\dot{\theta}\sin\theta \\ z = x\sin\theta &\Rightarrow \dot{z} = \dot{x}\sin\theta + x\dot{\theta}\cos\theta \end{aligned}\right\} \Rightarrow \dot{y}^2 + \dot{z}^2 = \dot{x}^2 + x^2\dot{\theta}^2$$



the total potential energy is

$$U = mgx\sin\theta \quad (1.2)$$

the total co-content energy is

$$J = \frac{1}{2} C\dot{\theta}^2 + \frac{1}{2} C' \dot{\phi}^2 + \frac{1}{2} B\dot{x}^2 \approx \frac{1}{2} C\dot{\theta}^2 \quad (1.3)$$

so

$$L = T - U = \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} m\left(\dot{x}^2 + x^2\dot{\theta}^2 + \frac{2}{5} r^2 \dot{\phi}^2\right) - mgx\sin\theta \quad (1.4)$$

assuming no slippery

$$dx = ds = rd\phi \quad (1.5)$$

then

$$L = T - U = \frac{1}{2}\left(I_\theta + mx^2\right)\dot{\theta}^2 + \frac{7}{10} m\dot{x}^2 - mgx\sin\theta \quad (1.6)$$

$$\underline{\underline{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial J}{\partial \dot{\theta}} = F_\theta}}$$

$$\frac{\partial L}{\partial \dot{\theta}} = I_\theta \dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(I_\theta + mx^2\right)\ddot{\theta} \quad (1.7)$$

$$\frac{\partial L}{\partial \theta} = -mgx\cos\theta \quad (1.8)$$

$$\frac{\partial J}{\partial \dot{\theta}} = C\dot{\theta} \quad (1.9)$$

$$\boxed{\left(I_\theta + mx^2\right)\ddot{\theta} + mgx\cos\theta + C\dot{\theta} = F_\theta} \quad (1.10)$$

$$\underline{\underline{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial J}{\partial \dot{x}} = F_x}}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{7}{5} m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{7}{5} m\ddot{x} \quad (1.11)$$

$$\frac{\partial L}{\partial x} = mx\dot{\theta}^2 - mg\sin\theta \tag{1.12}$$

$$\frac{\partial J}{\partial \dot{x}} = 0 \tag{1.13}$$

$$\frac{7}{5}\ddot{x} - x\dot{\theta}^2 + g\sin\theta = F_x = 0 \tag{1.14}$$

For small θ: $\sin\theta \approx \theta$, and with $g = 9.8\ m.s^{-2}$, linearize Eq.(1.14) to obtain

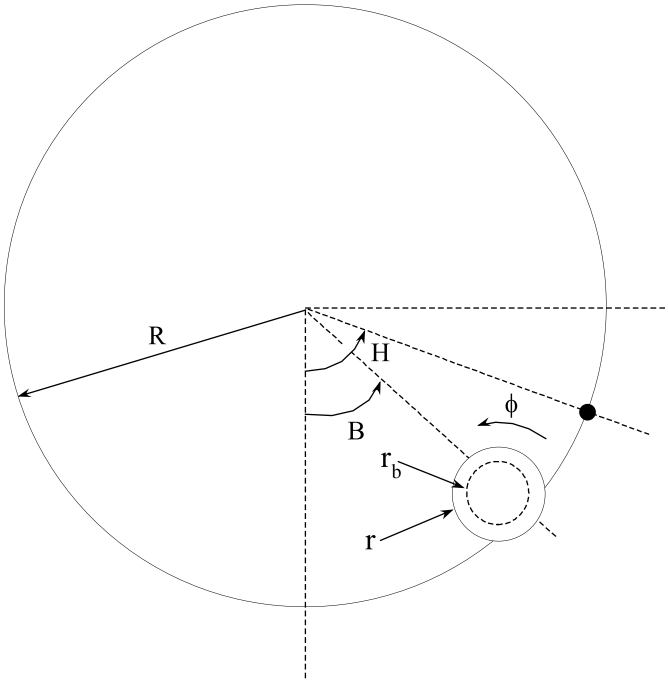
$$\ddot{x} + 7\theta = 0 \tag{1.15}$$

or

$$\frac{X(s)}{\Theta(s)} = \frac{-7}{s^2} \tag{1.16}$$

Ball and Hoop

Duy-Ky Nguyen Date 14-09-1993



Equation for the transformation of coordinates:

$$\phi = \frac{R}{r}(H - B) \tag{2.1}$$

Translational velocity of the Ball

$$v = (R - r)\dot{B}; \tag{2.2}$$

The hoop rotates only, but the ball rotates and translates as well. The total *co-kinetic energy* of the system

$$T = \frac{1}{2}I_a\dot{H}^2 + \frac{1}{2}I_b\dot{\phi}^2 + \frac{1}{2}M_b v^2$$

From (2.1) and (2.2)

$$T = \frac{1}{2}\left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - I_b\left(\frac{R}{r}\right)^2\dot{H}\dot{B} + \frac{1}{2}\left[I_b\left(\frac{R}{r}\right)^2 + M_b(R-r)^2\right]\dot{B}^2 \tag{2.3}$$

The total *potential energy* of the system:

$$U = M_b g.(R - r).(1 - \cos B) \tag{2.4}$$

For the ball and hoop, although the ball translates, there is no translational firction for the ball. The total *co-content energy* of the system

$$J = \frac{1}{2}B_m\dot{H}^2 + \frac{1}{2}B_b\dot{\phi}^2$$

From Eq.(2.1)

$$J = \frac{1}{2}\left[B_m + B_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - B_b\left(\frac{R}{r}\right)^2\dot{H}\dot{B} + \frac{1}{2}B_b\left(\frac{R}{r}\right)^2\dot{B}^2 \tag{2.5}$$

and

$$L = T - U = \frac{1}{2}\left[I_a + I_b\left(\frac{R}{r}\right)^2\right]\dot{H}^2 - I_b\left(\frac{R}{r}\right)^2\dot{H}\dot{B} + \frac{1}{2}\left[I_b\left(\frac{R}{r}\right)^2 + M_b(R-r)^2\right]\dot{B}^2 - M_b g.(R-r).(1-\cos B) \tag{2.6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{H}}\right)+\frac{\partial J}{\partial H}-\frac{\partial L}{\partial \dot{H}}=F_H=0$$

$$\frac{\partial L}{\partial \dot{H}}=\left[I_a+I_b\left(\frac{R}{r}\right)^2\right]\dot{H}-I_b\left(\frac{R}{r}\right)^2\dot{B}\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{H}}\right)=\left[I_a+I_b\left(\frac{R}{r}\right)^2\right]\ddot{H}-I_b\left(\frac{R}{r}\right)^2\ddot{B}$$

$$\frac{\partial L}{\partial H}=0$$

$$\frac{\partial J}{\partial \dot{H}}=\left[B_m+B_b\left(\frac{R}{r}\right)^2\right]\dot{H}-B_b\left(\frac{R}{r}\right)^2\dot{B}$$

$$\left[I_a+I_b\left(\frac{R}{r}\right)^2\right]\ddot{H}+\left[B_m+B_b\left(\frac{R}{r}\right)^2\right]\dot{H}-\left(\frac{R}{r}\right)^2\left(I_b\ddot{B}+B_b\dot{B}\right)=T \tag{2.7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{B}}\right)+\frac{\partial J}{\partial B}-\frac{\partial L}{\partial \dot{B}}=F_B=0$$

$$\frac{\partial L}{\partial \dot{B}}=-I_b\left(\frac{R}{r}\right)^2\dot{H}+\left[I_b\left(\frac{R}{r}\right)^2+M_b\left(R-r\right)^2\right]\dot{B}\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{B}}\right)=-I_b\left(\frac{R}{r}\right)^2\ddot{H}+\left[I_b\left(\frac{R}{r}\right)^2+M_b\left(R-r\right)^2\right]\ddot{B}$$

$$\frac{\partial L}{\partial B}=-M_b g.\left(R-r\right)\sin B$$

$$\frac{\partial J}{\partial \dot{B}}=-B_b\left(\frac{R}{r}\right)^2\dot{H}+B_b\left(\frac{R}{r}\right)^2\dot{B}$$

$$\left(\frac{R}{r}\right)^2\left(I_b\ddot{H}+B_b\dot{H}\right)-\left[I_b\left(\frac{R}{r}\right)^2+M_b\left(R-r\right)^2\right]\ddot{B}-B_b\left(\frac{R}{r}\right)^2\dot{B}-M_b g\left(R-r\right)\sin B=0 \tag{2.8}$$

Assume that

$$\begin{cases}\sin B\approx B\\ r\approx r_b\\ I_b\approx\frac{2}{5}M_b r_b^2\end{cases}$$

then from Eq.(2.8)

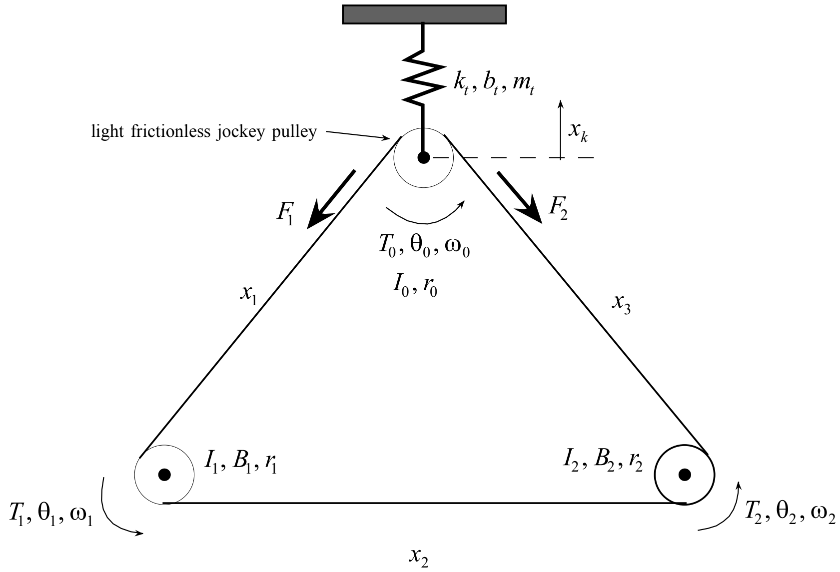
$$\left(\ddot{H}+\frac{B_b}{I_b}\dot{H}\right)-\frac{7}{2}\ddot{B}-\frac{B_b}{I_b}\dot{B}-\frac{5}{2R}gB=0$$

or

$$\frac{B(s)}{H(s)}=\frac{s\left(s+\frac{B_b}{I_b}\right)}{\frac{7}{2}s^2+\frac{B_b}{I_b}s+\frac{5}{2R}g} \tag{2.9}$$

Coupled Drives

Duy-Ky Nguyen **14-09-1993**



Because 3 pulleys rotate where the translational motion of the jockey pulley is negligible, so the total co-kinetic energy of the system is

$$T = \frac{1}{2} I_0 \dot{\theta}_0^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 = \frac{1}{2} I \left(\dot{\theta}_0^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 \right)$$

where

$$x_1 = r_0\theta_0 - r_1\theta_1 = r(\theta_0 - \theta_1)$$
$$x_2 = r_1\theta_1 - r_2\theta_2 = r(\theta_1 - \theta_2)$$
$$x_3 = r_2\theta_2 - r_3\theta_3 = r(\theta_2 - \theta_3)$$

and the total potentialenergy of the system is

$$U = \frac{1}{2} K\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2} Kr^2\left[(\theta_0 - \theta_1)^2 + (\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2\right] = \frac{1}{2} Kr^2\left(\theta_0^2 + \theta_1^2 + \theta_2^2 - \theta_0\theta_1 - \theta_1\theta_2 - \theta_2\theta_0\right)$$

where K : the extension coefficint of the belt.

Thus

$$L = T - U = \frac{1}{2} I\left(\dot{\theta}_0^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2} Kr^2\left(\theta_0^2 + \theta_1^2 + \theta_2^2 - \theta_0\theta_1 - \theta_1\theta_2 - \theta_2\theta_0\right)$$

Because the jockey pulley is light friction, so the total co-content energy of the system is

$$J = \frac{1}{2} B_1 \dot{\theta}_1^2 + \frac{1}{2} B_2 \dot{\theta}_2^2 = \frac{1}{2} B\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} + \frac{\partial J}{\partial \dot{\theta}_i} = F_i$$

$$i = 0 \;\Rightarrow\; I\theta_0^2 + Kr^2\left(2\theta_0 - \theta_1 - \theta_2\right) = 0 \qquad (3.1)$$
$$i = 1 \;\Rightarrow\; I\ddot{\theta}_1 + B\dot{\theta}_1 + Kr^2\left(2\theta_1 - \theta_2 - \theta_0\right) = T_1 \qquad (3.2)$$
$$i = 2 \Rightarrow\; I\ddot{\theta}_2 + B\dot{\theta}_2 + Kr^2\left(2\theta_2 - \theta_0 - \theta_1\right) = T_2 \qquad (3.3)$$

Taking the Laplace transform

$$\Theta_0\left(Is^2+2Kr^2\right)=Kr^2\left(\Theta_1+\Theta_2\right) \tag{3.4}$$

$$\Theta_1\left(I.s^2+B.s+2Kr^2\right)=T_1+Kr^2\left(\Theta_2+\Theta_0\right) \tag{3.5}$$

$$\Theta_2\left(I.s^2+B.s+2Kr^2\right)=T_2+Kr^2\left(\Theta_0+\Theta_1\right) \tag{3.6}$$

Adding Eqs.(3.5) and (3.6)

$$\Theta_1+\Theta_2=\frac{T_1+T_2+Kr^2\left(\Theta_1+\Theta_2\right)+2Kr^2\Theta_0}{I.s^2+B.s+2Kr^2}$$

by (3.4), and $\Omega_0=s.\Theta_0$

$$\Omega_0(s)=\frac{Kr^2}{I.s^3+BIs^2+3Kr^2Is+2BKr^2}\left(T_1+T_2\right) \tag{3.7}$$

Consider the jockey assembly, equating forces vertically:

$$2F\cos\alpha=m_t\ddot{x}_k+b_t\dot{x}_k+k_tx_k$$

$$X_k=\frac{2F\cos\alpha}{m_ts^2+b_ts+k_t} \tag{3.8}$$

assuming $F_1=F_2=F$ the tension in the belt.

then

$$T_0=\left(T_2-B\omega_2\right)-\left(T_1-B\omega_1\right)=\left(T_2-T_1\right)-B\left(\omega_1-\omega_2\right)$$

subtracting Eq.(3.6) from Eq.(3.5)

$$\Omega_1-\Omega_2=\frac{s\left(T_1-T_2\right)}{I.s^2+B.s+3Kr^2}$$

thus

$$T_0=\frac{\left(B.s-1\right)\left(T_1-T_2\right)}{I.s^2+B.s+3Kr^2}$$

from Eq.(3.8)

$$X_k(s)=\frac{2\cos\alpha\left(B.s-1\right)}{\left(m_ts^2+b_ts+k_t\right)\left(I.s^2+B.s+3Kr^2\right)}\left(T_1-T_2\right) \tag{3.9}$$

From Eqs.(3.7) and (3.9), we define a decoupler

$$\begin{cases}U_1=T_1+T_2\\U_2=T_1-T_2\end{cases}\Leftrightarrow\begin{cases}T_1=0.5U_1+0.5U_2\\T_2=0.5U_1-0.5U_2\end{cases} \tag{3.10}$$

then Eqs.(3.7) &(3.9) become

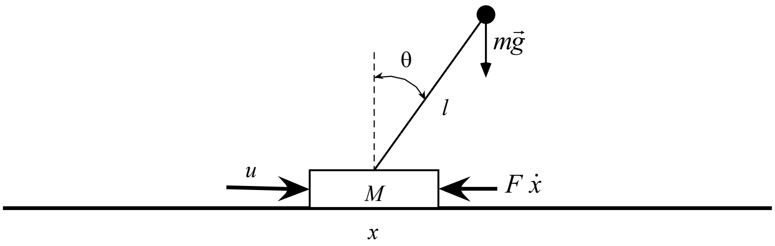
$$\Omega_0(s)=\frac{Kr^2}{I.s^3+BIs^2+3Kr^2Is+2BKr^2}U_1 \tag{3.11}$$

$$X_k(s)=\frac{2\cos\alpha\left(B.s-1\right)}{\left(m_ts^2+b_ts+k_t\right)\left(I.s^2+B.s+3Kr^2\right)}U_2 \tag{3.12}$$

Now, if we let $U_1=0$, we have the tension control, and if $U_2=0$ then speed control.

Inverted Pendulum

Duy-Ky Nguyen **11-03-95**



System kinetic energy is

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right) \quad (4.1)$$

system potential energy is

$$U = l\cos\theta \quad (4.2)$$

system co-content nergy is

$$J = 0 \quad (4.3)$$

We have the following coordinate relation

$$\begin{cases} x_p = x + l\sin\theta \;\Rightarrow\; \dot{x}_p = \dot{x} + l\dot{\theta}\cos\theta \\ y_p = l\cos\theta \;\Rightarrow\; \dot{y}_p = -l\dot{\theta}\sin\theta \end{cases} \quad (4.4)$$

so Eq.(4.3) becomes

$$T = \tfrac{1}{2}(m+M)\dot{x}^2 + \tfrac{1}{2}m\left(2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)$$

hence

$$L = T - U = \tfrac{1}{2}(m+M)\dot{x}^2 + \tfrac{1}{2}m\left(2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) - mgl\cos\theta \quad (4.5)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial J}{\partial \dot{x}} = F_x$$

$$\frac{\partial L}{\partial \dot{x}} = (m+M)\dot{x} + ml\dot{\theta}\cos\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m+M)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial J}{\partial \dot{x}} = 0$$

$$ml\cos\theta\,\ddot{\theta} + (m+M)\ddot{x} = u + ml\sin\theta\,\dot{\theta}^2 - F\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial J}{\partial \dot{\theta}} = F_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\dot{x}\cos\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta$$

$$\frac{\partial J}{\partial \dot{\theta}} = 0$$

$$l\ddot{\theta} + \cos\theta\,\ddot{x} = g\sin\theta$$

thus

$$\begin{cases} ml\cos\theta\,\ddot{\theta}+(m+M)\ddot{x}=u+ml\sin\theta\,\dot{\theta}^2-F\dot{x} \\ l\ddot{\theta}+\cos\theta\,\ddot{x}=g\sin\theta \end{cases}$$

so

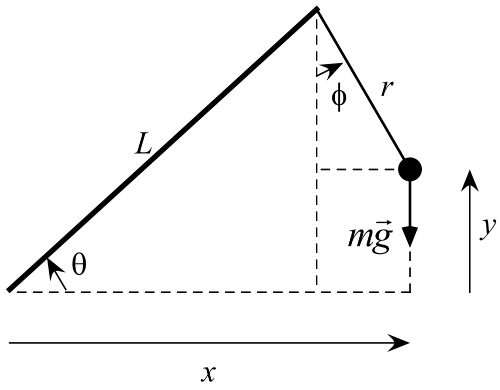
$$\begin{cases} -l\left(M+m\sin^2\theta\right)\sec\theta\,\ddot{\theta}=u+ml\sin\theta\,\dot{\theta}^2-F\dot{x}-(m+M)g\tan\theta \\ \left(M+m\sin^2\theta\right)\ddot{x}=u+ml\sin\theta\,\dot{\theta}^2-F\dot{x}-mg\sin\theta\cos\theta \end{cases} \tag{4.6}$$

or the linearized model

$$\begin{cases} \ddot{\theta}=-\dfrac{1}{Ml}u+\dfrac{g(m+M)}{Ml}\theta+\dfrac{F}{Ml}\dot{x} \\ \ddot{x}=\dfrac{1}{M}u-\dfrac{mg}{M}\theta-\dfrac{F}{M}\dot{x} \end{cases} \tag{4.7}$$

OFF-SHORE CRANE

Duy-Ky Nguyen 03-04-94



The total co-kinetic energy is

$$T = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) \tag{5.1}$$

where

$$\begin{cases} x = L\cos\theta + r\sin\phi \\ y = L\sin\theta - r\cos\phi \end{cases} \Rightarrow \begin{cases} \dot{x} = -L\dot{\theta}\sin\theta + r\dot{\phi}\cos\phi \\ \dot{y} = L\dot{\theta}\cos\theta + r\dot{\phi}\sin\phi \end{cases} \tag{5.2}$$

so

$$T = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\left[L^2\dot{\theta}^2 + r^2\dot{\phi}^2 - 2Lr\dot{\theta}\dot{\phi}\sin(\theta - \phi)\right] \tag{5.3}$$

the total potential energy is

$$U = mgr(1 - \cos\phi) \tag{5.4}$$

the total co-content energy is

$$J = \frac{1}{2} B_1\dot{\theta}^2 + \frac{1}{2} B_2\dot{\phi}^2 \tag{5.5}$$

so

$$L = T - U = \frac{1}{2}\left(I + mL^2\right)\dot{\theta}^2 + \frac{1}{2} mr^2\dot{\phi}^2 - mLr\dot{\theta}\dot{\phi}\sin(\theta - \phi) - mgr(1 - \cos\phi) \tag{5.6}$$

By the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial J}{\partial \dot{q}_i} = F_i \tag{5.7}$$

(1) $\underline{q_1 = \theta}$

$$\frac{\partial L}{\partial \dot{\theta}} = \left(I + mL^2\right)\dot{\theta} - mLr\dot{\phi}\sin(\theta - \phi)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(I + mL^2\right)\ddot{\theta} - mLr\ddot{\phi}\sin(\theta - \phi) - mLr\dot{\phi}\left(\dot{\theta} - \dot{\phi}\right)\cos(\theta - \phi)$$

$$\frac{\partial L}{\partial \theta} = -mLr\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$

$$\frac{\partial J}{\partial \dot{\theta}} = B_1\dot{\theta}$$

$$\boxed{\left(I + mL^2\right)\ddot{\theta} - mLr\ddot{\phi}\sin(\theta - \phi) + B_1\dot{\theta} + mLr\dot{\phi}^2\cos(\theta - \phi) = F_\theta} \tag{5.8}$$

(2) $\underline{q_2 = \phi}$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} - mLr\dot{\theta}\sin(\theta-\phi)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi} - mLr\ddot{\theta}\sin(\theta-\phi) - mLr\dot{\theta}(\dot{\theta}-\dot{\phi})\cos(\theta-\phi)$$

$$\frac{\partial L}{\partial \phi} = mLr\dot{\theta}\dot{\phi}\cos(\theta-\phi) - mgr\sin\phi$$

$$\frac{\partial J}{\partial \dot{\phi}} = B_2\dot{\phi}$$

$$mr^2\ddot{\phi} + B_2\dot{\phi} + mgr\sin\phi - mLr\ddot{\theta}\sin(\theta-\phi) - mLr\dot{\theta}^2\cos(\theta-\phi) = F_\phi \quad (5.9)$$

Let

$$\theta = \theta_0 + \psi \quad (5.10)$$

Assume

$$B_2 = 0 \quad (5.11)$$

with

$$F_\phi = 0 \quad (5.12)$$

Using

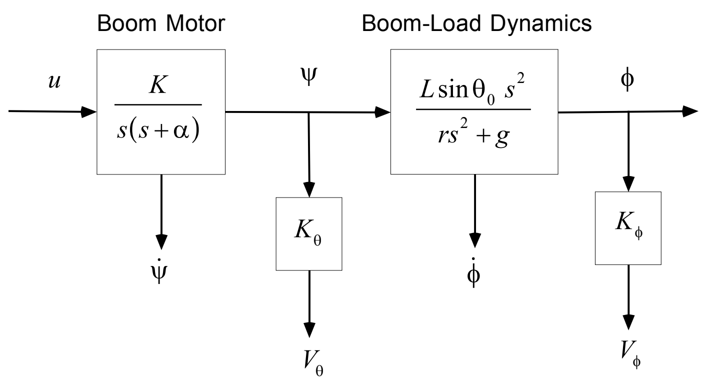
$$\sin(\theta-\phi) = \sin[\theta_0 + (\psi-\phi)] = \sin\theta_0\cos(\psi-\phi) + \cos\theta_0\sin(\psi-\phi) \quad (5.13)$$

$$\cos(\theta-\phi) = \cos[\theta_0 + (\psi-\phi)] = \cos\theta_0\cos(\psi-\phi) - \sin\theta_0\sin(\psi-\phi) \quad (5.14)$$

we can linearize Eq.(9) at $\psi = 0$ and $\phi = 0$ to get

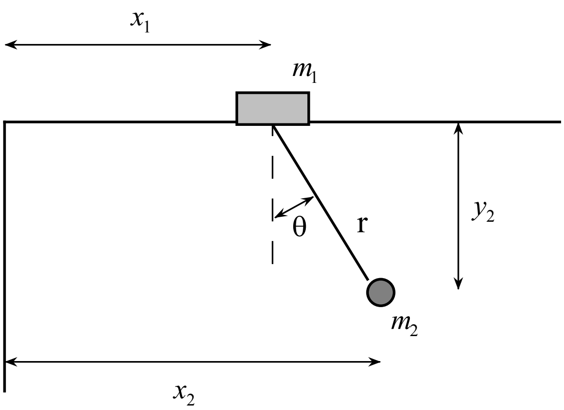
$$\frac{\Phi(s)}{\Psi(s)} = \frac{L\sin\theta_0\, s^2}{rs^2 + g} \quad (5.15)$$

Thus we have the following system block diagram



OVER-HEAD CRANE

Duy-Ky Nguyen **03-04-1994**



The total co-kinetic energy is

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right) \tag{6.1}$$

$$\begin{cases} x_2 = x_1 + r \sin\theta \\ y_2 = r\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = \dot{x}_1 + r\dot{\theta}\cos\theta \\ \dot{y}_2 = -r\dot{\theta}\sin\theta \end{cases} \tag{6.2}$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 + m_2 \dot{x}_1 r \dot{\theta}\cos\theta \tag{6.3}$$

the total potential energy is

$$U = m_2 g r (1 - \cos\theta) \tag{6.4}$$

the total co-content energy is

$$J = \frac{1}{2} B_1 \dot{x}_1^2 \tag{6.5}$$

so

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 + m_2 \dot{x}_1 r \dot{\theta}\cos\theta - m_2 g r (1 - \cos\theta) \tag{6.6}$$

By the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial J}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \tag{6.7}$$

(1) $\underline{q_1 = x_1}$

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + m_2 r \dot{\theta}\cos\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1} \right) = (m_1 + m_2)\ddot{x}_1$$

$$\frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial J}{\partial \dot{x}_1} = B_1 \dot{x}_1$$

$$(m_1 + m_2)\ddot{x}_1 + B_1 \dot{x}_1 = F_{x_1} \tag{6.8}$$

(2) $\underline{q_2 = \theta}$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 r^2 \dot{\theta} + m_2 \dot{x}_1 r \cos\theta \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 r^2 \ddot{\theta} + m_2 \ddot{x}_1 r \cos\theta - m_2 \dot{x}_1 r \dot{\theta} \sin\theta$$

$$\frac{\partial L}{\partial \theta} = -m_2 r \left(\dot{x}_1 \dot{\theta} + g\right) \sin\theta$$

$$\frac{\partial J}{\partial \dot{\theta}} = 0$$

$$m_2 r^2 \ddot{\theta} + m_2 \ddot{x}_1 r \cos\theta + m_2 g r \sin\theta = 0 \qquad (6.9)$$

By

$$\cos\theta \approx 1, \qquad \sin\theta \approx \theta$$

then from Eq.(6.9)

$$\frac{\Theta(s)}{X_1(s)} = \frac{-s^2}{rs^2 + g} \qquad (6.10)$$