Digital Control System: A Precise Approach

Duy-Ky Nguyen **26-08-1996**

Nowadays, most controllers are implemented as digital controllers rather than analog ones since digital controllers are much more flexible and they can be used to implement highly complicated control functions. Simple control functions can be implemented with analog controllers, however they are less reliable due to the drifting of op-amps, the leakage of capacitors and the aging of analog components such as resistors, capacitors etc.

The z-transform (Section 5) and a discretization of a state-space model (Section 7) will be used in the latter sections on discrete-time SMC. Although these results are well-known in the digital control literature, this section is included to review digital control systems in a compact, concise and logical manner. We first start with a sampling process to convert an analog signal into digital signal in deriving the *starred transform* which provides a basis to develop the *Z-transform*. The Z-transform will then define a mapping between the continuous-time s-plane and the discrete-time z-plane as a mathematical tool to design a digital controller, as the Laplace transform is used to design an analog controller.

1. Some Terminology

In this subsection, we are concerned with the difference between *discrete-time* systems and *digital* systems whereas they both are discretized from an *analog* system. In the real world, all signals are *analog signals* whose amplitudes take *infinite* values within some ranges. Their values can be any real numbers with fraction and the difference between successive values is *infinitesimal*.

Analog signals are continuous-time signals. *Discrete-time signals*, or *discrete signals* for short, take finite values of any real numbers, hence the difference of successive values is finite and is a real number.

A digital signal is a subset of discrete-time signals. A values of a digital signal must be an *integer*, thus the difference of successive values is finite and is an integer. These integers are within the range determined by the length of registers in a ADC, the range is $\left[0 \sim 2^n\right]$or $\left[-2^{n-1} \sim 2^{n-1}-1\right]$ for a n-bit ADC.

Discrete-time control theory is used to design a discrete-time controller which is employed to implement a digital controller for a digital control system.

A continuous-time system is discretized using discrete-time control theory, there is no digitization of a system since it is a special case of discretization as a real number is rounded off to an integer. An analog signal is digitized, also known as quantized, by an ADC. There is no discretization of an analog signal since there is no such converter available nowadays.

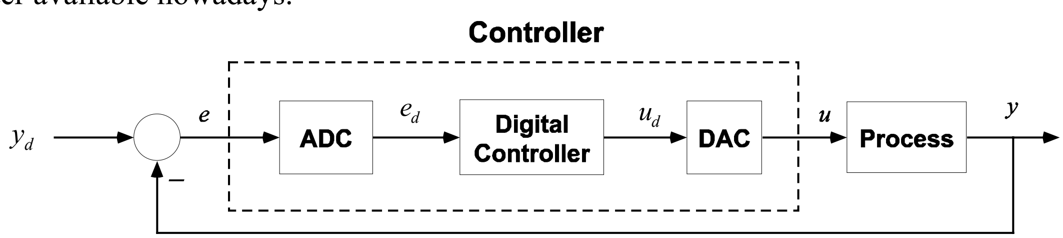
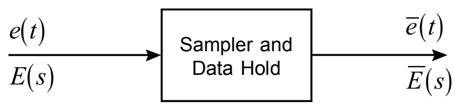


Fig. 1: Digital Control System

where $y, y_d, e,$ and u are analog signals while e_d and u_d are digital signals. ADC and DAC *are analog-to-digital* and *digital-to-analog* converters, respectively. Digital controller can be either a micro-controller or a digital signal processor (DSP) where ADC and DAC are on-chip (internal) or a general purpose computer with external ADC/DAC. For 8-bit processors such as 8051 Intel family or 68HC11 Motorola family, ADC and DAC are also 8-bit. However, higher resolution ADC's are expensive, so that 10-bit or 12-bit ADC are usually used for 16-bit processors such as 80196 Intel family, 68000 Motorola family, 32-bit 486-PC, 64-bit Pentium-PC.

2. ADC as Physical Sampler

Practically, ADC is a *zero-order-hold* sampler of *successive approximation* type



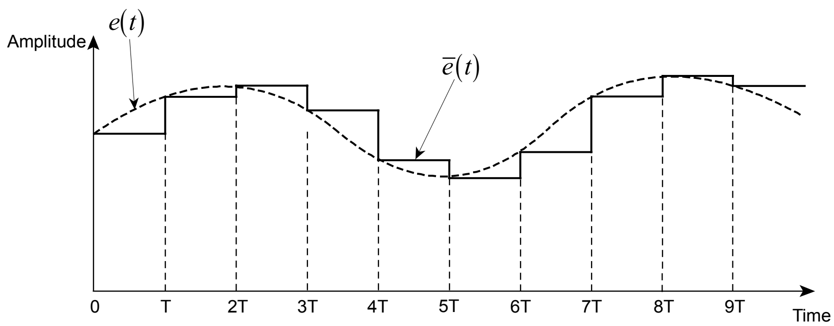


Fig. 2: Sample and Hold Signal

The *sampled and zero hold signal* $\bar{e}(t)$ can be mathematically expressed as

$$\bar{e}(t) = \sum_{k=-\infty}^{\infty} e(kT)\{\mathscr{U}(t-kT) - \mathscr{U}[t-(k+1)T]\} \tag{1}$$

where $\mathscr{U}(t)$ is the unit step function, that is

$$\mathscr{U}(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases} \tag{2}$$

Using the shifting theorem of Laplace transform gives

$$\bar{E}(s) = \sum_{k=-\infty}^{\infty} e(kT)\left\{\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right\} = \left(\frac{1-e^{-Ts}}{s}\right)\sum_{k=-\infty}^{\infty}\left[e(kT)e^{-kTs}\right] \tag{3}$$

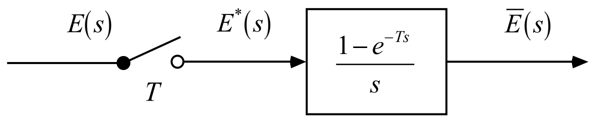
3. Starred Transform as Ideal Sampler

On the basis of Eq.(3), the *starred transform* is defined as

$$E^*(s) = \sum_{k=-\infty}^{\infty} e(kT)e^{-kTs} \tag{4}$$

then Eq.(3) can be written as

$$\bar{E}(s) = \left(\frac{1-e^{-Ts}}{s}\right)E^*(s) \tag{5}$$



Eq.(4) gives the inverse Laplace transform of $E^*(s)$ as

$$e^*(t) = \mathscr{L}^{-1}\{E^*(s)\} = \sum_{k=-\infty}^{\infty}[e(kT)\delta(t-kT)] \quad (6)$$

where $\delta(t)$ is a Dirac delta function.

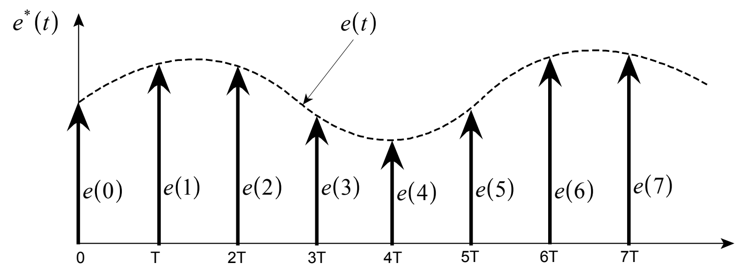


Fig. 3: A Representation of $e^*(t)$

4. Sampling Characteristics and Signal Reconstruction

This subsection founds a basis for choosing sampling rate and necessity of an analog low-pass filter as an anti-aliasing filter before ADC. Sampling characteristics and signal reconstruction is determined by the following theorem

Theorem 1: Laplace Transform of Sampled Signal

An alternative expression for the starred transform is given by

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s + jk\omega_s) \quad (7)$$

thus the Laplace transform of a sampled signal is equal to an infinite sum of Laplace transform of the Laplace transform of its original signal. The s-plane may be divided into a *primary strip* and *complementary strips* as shown in Fig. 4.

Proof

Following the proof in Lockhart *et al.* 1989, let

$$\mathscr{S}(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT) \quad (8)$$

then Eq.(6) can be written as

$$e^*(t) = e(t).\mathscr{S}(t) \quad (9)$$

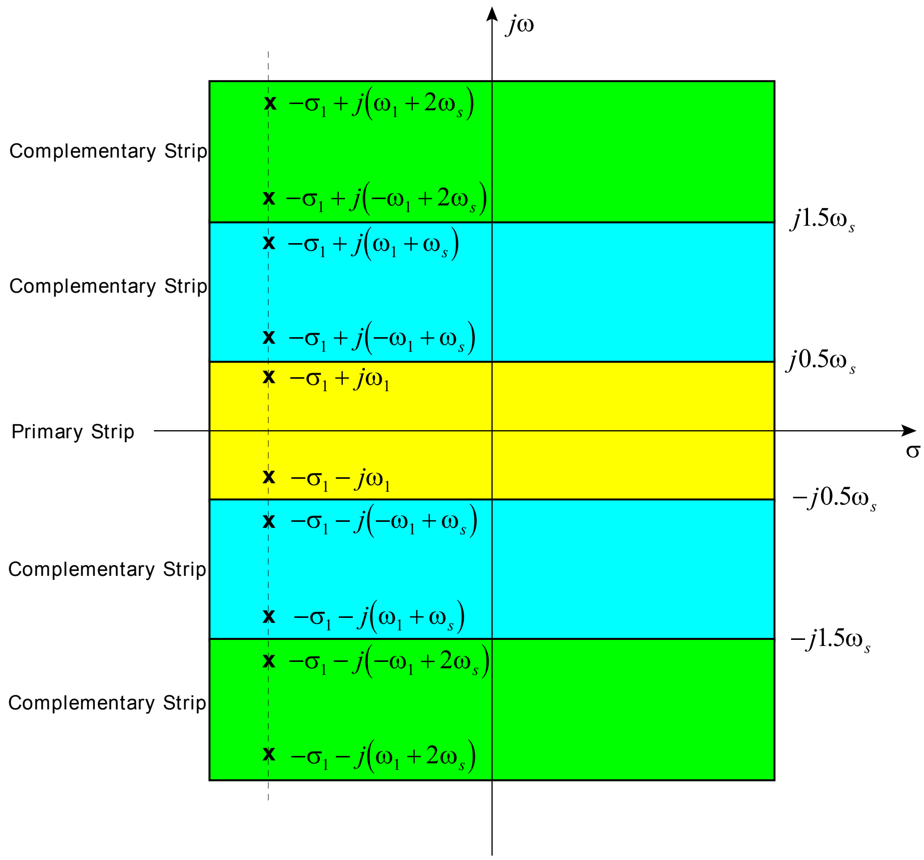


Fig. 4: Primary and Complementary Strips

Since $\mathscr{S}(t)$ is a periodic function $\delta(t)$ whose period is *T*, its Fourrier series is

$$\mathscr{S}(t)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\int_0^T \delta(t)e^{-j2\pi k\tau/T}d\tau\right)e^{j2\pi kt/T}=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{jk\omega_s t} \tag{10}$$

where

$$\omega_s=\frac{2\pi}{T} \tag{11}$$

thus Eq.(9) can be read as

$$e^*(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e(t)e^{jk\omega_s t} \tag{12}$$

and its Laplace transform is

$$E^*(s)=\mathscr{L}\{e^*(t)\}=\int_0^{\infty}\frac{1}{T}\sum_{k=-\infty}^{\infty}e(t)e^{-(s-jk\omega_s)t}dt=\frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^{\infty}e(t)e^{-(s-jk\omega_s)t}dt=\frac{1}{T}\sum_{k=-\infty}^{\infty}E(s+jk\omega_s)$$

Q.E.D.

4.1. Signal Reconstruction

To see the significance of Theorem 1, consider the case of frequency response, when $s=j\omega$ Eq.(7) gives

$$E^*(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}E[j(\omega+k\omega_s)] \tag{13}$$

thus the spectrum of sampled signal consists of an infinite number of replicas of the analog signal spectrum scaled by the factor $1/T$ and the frequency shifted by multiples of ω_s.

If the sampling time T is low enough such that $\omega_{\max} < \omega_s/2$, then the sampled spectrum retains the shape of the analog one and the analog signal can be reconstructed from its sampled spectrum within the band $[-\omega_s/2, \omega_s/2]$.

For signal reconstruction, the sampling rate must satisfy the sampling theorem, also known as Shannon's Theorem which states that the sampling frequency ω_s must be at least twice the value of the highest significant frequency in the signal. Since an ideal low-pass reconstruction filter cannot be implemented, one rule of thumb is to choose T as one-tenth of the smallest process time constant or the desired closed-loop time constant. Another convenient rule suggests sampling at the rate of 6 to 10 times per cycle.

Proposition 1: Choice of Sampling Time
In this work, the sampling time is chosen such that $\omega_s \geq 10\omega_{\max}$. With this choice, ignoring the sampling effect will introduce a maximum tolerance of gain about 0.12% and of phase about 3^0.

Proof:

We have

$$e^{-sT} = \frac{e^{-sT/2}}{e^{sT/2}} = \frac{1 - \frac{sT}{2} + \frac{(sT)^2}{2^2 2!} - \cdots}{1 + \frac{sT}{2} + \frac{(sT)^2}{2^2 2!} + \cdots} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}$$

then Eq.(5) gives

$$G_h(s) = \frac{1 - e^{-sT}}{s} \approx \frac{1}{s}\left(1 - \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}\right) = \frac{T}{\frac{T}{2}s + 1}$$

Since the overall DC gain will be determined at the final design stage, the sampling time will be discarded to have

$$G_h(s) \approx \frac{1}{\frac{T}{2}s + 1}$$

thus ignoring the sampling effect will introduce a maximum tolerance of

$$G_h(j\omega_{\max}) \approx \frac{1}{1 + j\frac{0.1}{2}} = \frac{1}{1 + j0.05} = 0.9988\angle -2.86^0$$

Q.E.D

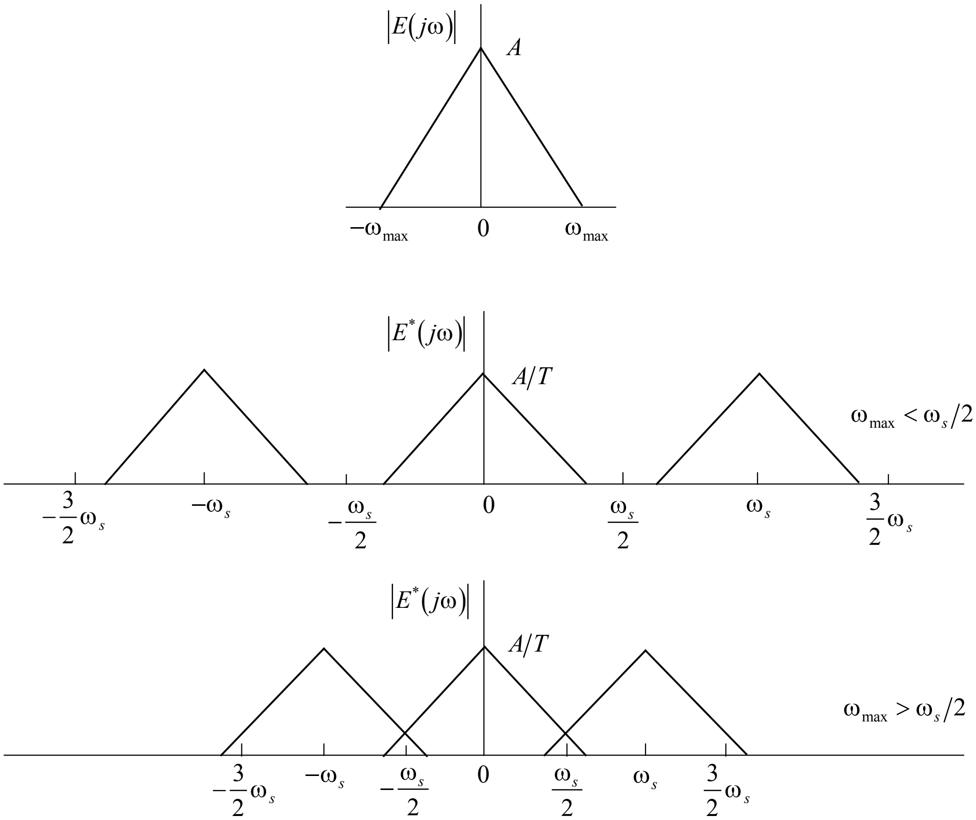


Fig. 5: Spectrum of Analog and Sampled Signal

4.2. Aliasing or Folding

Eq.(7) reveals that if $E(s)$ has a pole at p_0, then $E^*(s)$ will have poles at $(p_0 + jn\omega_s)$ where $n = 0, \pm 1, \pm 2, \cdots$ For example, suppose that $E(s)$ has one pair of poles located at $-a \pm j0.6\omega_s$, outside the primary strip. The sampling will then "fold" this pair back into the primary strip, to the location $-a \pm j0.4\omega_s$. So $e^*(t)$ contains components inside the primary strip that do not occur in $e(t)$. This property applies for poles only, not for zeros, since poles of a transfer function component in Eq.(7) will be poles of the whole transfer function; however, zeros of a component are not necessarily the zeros of the whole.

An anti-aliasing analog filter is used to solve this problem by filtering out all noise. It is a low-pass filter with the cut-off frequency is based on the highest frequency of the analog signal.

5. Z-Transform

On the basis of Eq.(4), the *Z transform* is defined as

$$E(z) = \mathcal{Z}\{e(k)\} = \sum_{k=-\infty}^{\infty} e(k) z^{-k} \tag{14}$$

where the mapping between the s-plane and the *z*-plane is defined as

$$z = e^{Ts} \Leftrightarrow s = \frac{1}{T}\ln(z) \tag{15}$$

since comparing Eqs.(4) and (14) yields

$$E^*(s) = E(z)\big|_{z=e^{Ts}} \tag{16}$$

The Laplace transform is used in obtaining a *control transfer function* for implementing a analog controller. We will see that the Z-transform will be employed in obtaining a *control difference equation* for implementing a digital controller.

Two of the most important properties of the Z-transform are the linearity and the real are given below

$$\mathcal{Z}\{ae_1(k)+be_2(k)\} = a\mathcal{Z}\{e_1(k)\}+b\mathcal{Z}\{e_2(k)\} \tag{17}$$

and

$$\mathcal{Z}\{e(k-n)\} = z^{-n}\mathcal{Z}\{e(k)\} \tag{18}$$

where n is a positive integer.

Since by the definition of the Z-transform, we have

$$\mathcal{Z}\{ae_1(k)+be_2(k)\} = \sum_{k=-\infty}^{\infty}[ae_1(k)+be_2(k)]z^{-k} = a\sum_{k=-\infty}^{\infty}e_1(k)z^{-k} + b\sum_{k=-\infty}^{\infty}e_2(k)z^{-k} = a\mathcal{Z}\{e_1(t)\}+b\mathcal{Z}\{e_2(t)\}$$

and

$$\mathcal{Z}\{e(k-n)\} = \sum_{k=-\infty}^{\infty}[e(k-n)]z^{-k} = \sum_{m=-\infty}^{\infty}e(m)z^{-(n+m)} = z^{-n}\sum_{k=-\infty}^{\infty}e(m)z^{-m} = z^{-n}\mathcal{Z}\{e(k)\}$$

where $m = k - n$.

6. Some Typical S-Z Transformations

For completeness, this subsection presents a brief derivation of 2 well-known transformations: backward difference method and bilinear method. The Z-transform defines the mapping in Eq.(15) as the exact transformation between the *s*-plane and the *z*-plane. This transform should be used to convert a continuous-time system into a discrete-time one. A direct substituting Eq.(15) into a transfer function in *s*-plane will produce an infinite difference equation due to the Taylor expansion of the natural logarithmic function. This necessitates an approximate transformation to obtain finite difference equations if the substitution method is used *in place* of the *z*-transform. From a single term *s,* we can have *s* and $1/s$ corresponding to a differentiator and an integrator, respectively.

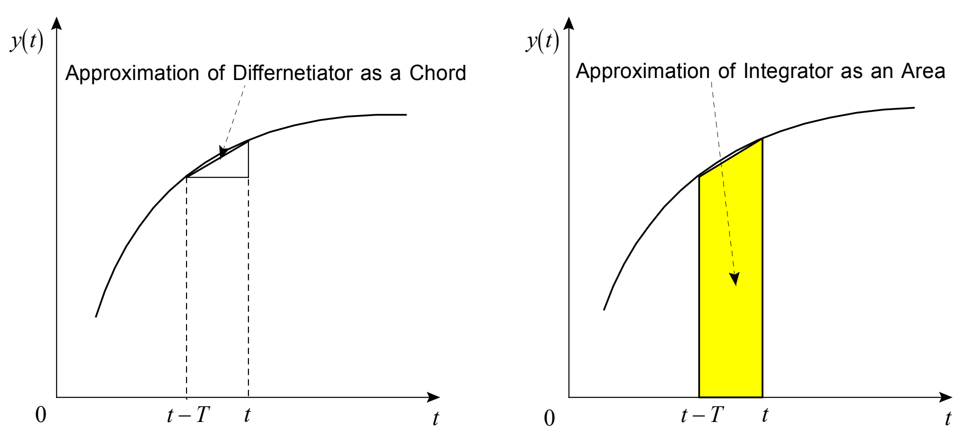


Fig. 6: Approximate Transformations for Differentiator and Integrator

The backward difference transformation approximates a differentiator as the slope of a chord

$$\frac{d}{dt}y(t) \approx \frac{y(t)-y(t-T)}{T}$$

taking Laplace transform gives

$$sY(s) \approx \frac{Y(s)-e^{-Ts}Y(s)}{T}$$

so

$$s \approx \frac{1-e^{-Ts}}{T} = \frac{1-z^{-1}}{T} \Leftrightarrow z \approx \frac{1}{1-Ts} \tag{19}$$

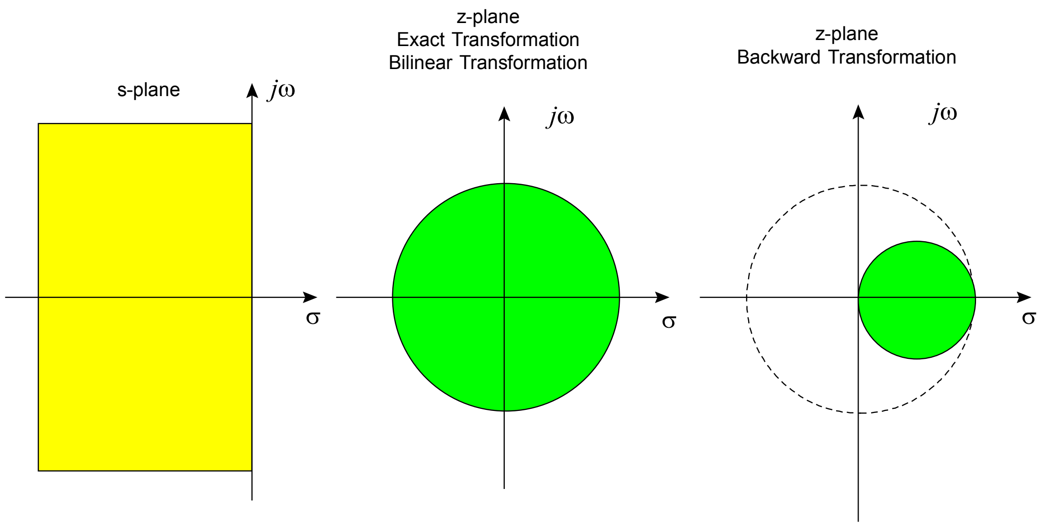


Fig. 7: Stability Regions

The bilinear transformation approximates an integrator as the area of a trapezoidal

$$\int_0^t y(\tau).d\tau - \int_0^{t-T} y(\tau).d\tau \approx \frac{T}{2}\left[y(t-T) + y(t)\right]$$

taking Laplace transform gives

$$\frac{1}{s}Y(s) - \frac{1}{s}e^{-Ts}Y(s) \approx \frac{T}{2}\left[e^{-Ts}Y(s) + Y(s)\right]$$

thus

$$\frac{1}{s} \approx \frac{T}{2}\frac{1+e^{-Ts}}{1-e^{-Ts}} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \Leftrightarrow z \approx \frac{(T/2)+s}{(T/2)-s} \tag{20}$$

The stability region in the s-plane is the RHP, to find its mapping in the z-plane, substituting the frequency contour $s = j\omega$ into Eqs.(15), (20) and (21) produces

$$z = e^{j\omega T} \tag{21}$$

$$z = \frac{1}{1-j\omega T} = \frac{1}{2} + \frac{1}{2}\frac{1+j\omega T}{1-j\omega T} = \frac{1}{2} + \frac{1}{2}e^{j\theta}, \theta = 2\tan^{-1}(2\omega T) \tag{22}$$

$$z \approx \frac{(T/2)+j\omega T}{(T/2)-j\omega T} = e^{j\theta}, \theta = 2\tan^{-1}(2\omega) \tag{23}$$

Thus a stable controller could be unstable under the backward difference transformation. This is a limitation of this transformation.

7. Discretization of Continuous-Time State-Space Equations

The z-transform is used to discretize a continuous-time transfer-function model for designing a discrete-time controller in the z-domain. Alternatively, a transfer-function model can be modified by taking into account the sampling process, and a continuous-time controller can be designed in the s-plane then discretized to obtain a discrete-time controller. To discretize a continuous-time state-space model, from the *digital control literature*, we have the following theorem:

Theorem 2: Discretization of State-Space Model

Consider the continuous-time state equation and output equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \tag{24}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{25}$$

under the assumption of zero-order hold that

$$\mathbf{u}(t) = \mathbf{u}(kT), kT \le t \le (k+1)T \quad . \tag{26}$$

Then a discrete-time representation of Eq.(24) will take the form

$$\mathbf{x}[(k+1)T] = \mathbf{A}_d(T)\mathbf{x}(kT) + \mathbf{B}_d(T)\mathbf{u}(kT) \tag{27}$$

$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT) \tag{28}$$

where

$$\mathbf{A}_d(T) = e^{\mathbf{A}_c T} \tag{29}$$

and

$$\mathbf{B}_d(T) = \left(\int_0^T e^{\mathbf{A}_c t}dt\right)\mathbf{B}_c \tag{30}$$

with T is a constant of sampling time, $\mathbf{A}_d(T) and \mathbf{B}_d(T)$ are thus constant matrices.

Proof

Following the proof in Ogata 1987, the Laplace transform of Eq.(24) give

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}_c\mathbf{X}(s) + \mathbf{B}_c\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A}_c)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}_c\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c\mathbf{U}(s) \tag{31}$$

Note that

$$(s\mathbf{I} - \mathbf{A}_c)^{-1} = \sum_{n=0}^{\infty}\frac{\mathbf{A}_c^n}{s^{n+1}}, \mathbf{A}_c^0 = \mathbf{I} \tag{32}$$

hence its inverse Laplace transform is

$$\mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A}_c)^{-1}\right\} = \sum_{n=0}^{\infty}\frac{(\mathbf{A}_c t)^n}{n!} = e^{\mathbf{A}_c t}, \qquad 0! = 1 \tag{33}$$

thus Eq.(32) becomes

$$\mathbf{X}(s) = e^{\mathbf{A}_c t}\mathbf{x}(0) + e^{\mathbf{A}_c t}\mathbf{B}_c\mathbf{U}(s) \tag{34}$$

then its inverse Laplace transform is

$$\mathbf{x}(t) = e^{\mathbf{A}_c t}\mathbf{x}(0) + e^{\mathbf{A}_c t}\int_0^t e^{-\mathbf{A}_c \tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{35}$$

This equation gives

$$\mathbf{x}[(k+1)T] = e^{\mathbf{A}_c(k+1)T}\mathbf{x}(0) + e^{\mathbf{A}_c(k+1)T}\int_0^{(k+1)T} e^{-\mathbf{A}_c\tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{36}$$

and

$$\mathbf{x}(kT) = e^{\mathbf{A}_c kT}\mathbf{x}(0) + e^{\mathbf{A}_c kT}\int_0^{kT} e^{-\mathbf{A}_c\tau}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{37}$$

Multiplying Eq.(37) by $e^{\mathbf{A}_c T}$ and subtracting from Eqs.(26) and (36) yields

$$\mathbf{x}[(k+1)T] = e^{\mathbf{A}_c T}\mathbf{x}(kT) + e^{\mathbf{A}_c(k+1)T}\int_{kT}^{(k+1)T} e^{-\mathbf{A}_c\tau}\mathbf{B}_c\mathbf{u}(kT)d\tau$$

then

$$\mathbf{x}[(k+1)T] == e^{\mathbf{A}_c T}\mathbf{x}(kT) + \left(\int_0^T e^{\mathbf{A}_c t}\mathbf{B}_c dt\right)\mathbf{u}(kT), t = (k+1)T - \tau \tag{38}$$

Substituting $t = kT$ in Eq.(25) gives Eq.(28).

Q.E.D.