PainlessCompact and Comprehensive Caculus

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Cacculus is backbone in Science &technology

My working phylosophy is the simplest is the best and the hardest to achieve

It's the best as it takes the lea st effort to complete and to maitain and it's the hardest as it require a deep aware of the subject in order to remove unnecessaies and keep only the minimum core in the simplest possible approach

Somedones try their his best to do thing as much complicated as possible just for their eggo that they could do a very unusual complicated thing

I see it's very costly in debuging in maintaining and to be upgraded with new bug fixed and new feature

This Calculus notes starts with Limit for main derivatives [product integer power ⇒ **Taylor series**

We start with limit as a basis for definition of derivative

By definition of derivative we can get main derivatives derivative of sum of functions and product with a const not product of 2 functions defered later after logarithmic as multiply ⇒ add easier to differentiating

No section of integral as we have to new approach than literature

The logarithmic and exponential function based on derivative and to complete derivative of power of fraction

Trigonometic Derivatives based on **Euler identity** a lot simpler than the way in current literature

Numerical methodhas provided a much simpler to solve differential Eq based on Taylor series a lot simpler than current literature using Kunte gutta and the likes

Logarithmic function is defined from derivative Exponential function is defined as inverserse func of log

We then formulate derivative of product and quotient of 2 func

Derivatives of sin and cos are not based on limit but on Euler formula By the way we revisit sin cos identities uing this very formula

Laplace transform is a powerful tool to sove differential eqs but we don't use Inver Laplace transform bu ather to use numerical method

I have an article namly **Clock Termination** using numerical method in improve clock quality for product at my work place at Symmetricom

Using numerical method to get result in form of graph plot rather function with Inverse Laplace transform to get However it's hard to see the result impact impact with function. to see the impactsome how we have to do 1 more step for function analysis so it's simpler to use numerical method using MLab laguage with Octave SW , an absolutely free while MLab cost few thoussands for license and Octave appears to me a lot better in an section of numerical method included in this note

1. Limit

Limit is back bone of Calculus

$\lim_{x \to a} f(x) = f(a+\varepsilon), \exists \varepsilon$ There exist a a very small number close to zero ε

The realreason is it's not allowed to do divide a zero but it's fine using limit

$$\lim_{x \to 0+} \left(\frac{1}{x}\right) = +\infty$$

$$\lim_{x \to 0-} \left(\frac{1}{x}\right) = -\infty$$

1.1. Limit Definitions

Per limit definition $\underset{x \to a}{f(x)} \equiv f(a)$

1.2. Limit properties

By Limit definition

Limit[expression of some f'sf]=expressionof each limit[f] invividually

Provided existence of lim

$$\lim[a * f \pm b * g] = * \lim[f] \pm b * \lim[g]$$

$$\lim\{f * g\} = \lim\{f\} * \lim\{g\}$$

$$\lim[f / g] = \lim[f] / \lim[g], \lim[g] \neq 0$$

$$\lim[cf] = c \lim[f], const\, c$$

$$\lim\left[\frac{f}{g}\right]\frac{\lim[f]}{\lim[g]}, \lim[g] \neq 0$$

$$\lim_{x \to a}[f(x)] \equiv f(a + \varepsilon), \exists \varepsilon \approx 0$$

1.3. Left&Right Limit

$$\lim_{x \to 0^+}\left[\frac{1}{x}\right] \equiv +\infty$$

$$\lim_{x \to 0^-}\left[\frac{1}{x}\right] \equiv -\infty$$

Limit does exist iff[if and only if]left $\lim_{a} = \lim_{a^+} = \lim_{a^-}$ right limit the same

1.4. Derivative

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\frac{f(x+h) - f(x)}{h} = 2x + h$$

$$hence$$

$$[x^2]^{(\ )} = 2x$$

$$f(x) = x^2$$

$$f(x+h) = (x+h)^2$$

$$f(x+h) - f(x) = (x+h)^2 - x^2 = h(x+h+x)$$

$$with\, a^2 - b^2 = (a-b)(a+b)$$

$$so$$

$$\frac{f(x+h) - f(x)}{h} = (x+h+x) \xrightarrow[h \to 0]{} 2x$$

$f(x) = x^n$

by

$$f(x+h) = (x+h)^n = x^n + n * h * x^{n-1} + h * (\cdots)$$

$$f(x+h) - f(x) = (x+h)^3 - x^3 = h\left[nx^{n-1} + h * (\cdots)\right]$$

so

nx^{n-1}

1.5. L'Hospital's Rule and Indeterminate Forms

1.5.1. ndeterminate Forms $\frac{0}{0} or \frac{\infty}{\infty}$

1.5.2. L'Hospital's Rule

$$L\left[\frac{f(x)}{g(x)}\right] = L\left[\frac{f(x)}{g(x)}\right] = \frac{-L[f]}{-L[g]} = \frac{L[-f]}{L[-g]} = \frac{L\left[f^+ - f^+ - f\right]}{L\left[g^+ - g^+ - g\right]} = \frac{L\left[f^+ - f\right] - L\left[f^+\right]}{L\left[g^+ - g\right] - L\left[g^+\right]} = \frac{L\left[f^+ - f\right] - L\left[f^+\right]}{L\left[g^+ - g\right] - L\left[g^+\right]}$$

$$= \frac{L\left[f^+ - f\right] - L\left[f^+\right]}{L\left[g^+ - g\right] - L\left[g^+\right]} = \frac{\frac{L\left[f^+ - f\right]}{L[h]} - \frac{L\left[f^+\right]}{L[h]}}{\frac{L\left[g^+ - g\right]}{L[h]} - \frac{L\left[g^+\right]}{L[h]}}$$

$As\ h \to 0$

$$\frac{L\left[f^+ - f\right]}{L[h]} = f'$$

$$\frac{L\left[f^+\right]}{L[h]} = 0$$

$Hence$

$$L\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$$

Example 1

$$\lim_{x \to b}\left[a\frac{x-b}{2x-2b}\right] = a\left[\frac{1}{2}\right]$$

We have

$$\lim_{x \to b}\left[a\frac{x-b}{2x-2b}\right] = \lim_{x \to b}\left[a\frac{x-b}{2(x-b)}\right] = \left[a\frac{1}{2}\right], x \neq b$$

2. Derivative & Differentiation

2.1. composite function

$$f(g(x)) = [f \circ g](x)$$

Let

$$f(x) = x^2$$

$$g(x) = x - 3$$

$$[f \circ g](x) = f(g(x)) = g^2(x) = (x-3)$$

2.2. Chain Rule

It's based on Limit properiesLimit of expressionis expression of limitsof individual

$$\frac{df}{dx} = \frac{df(u)}{du}\frac{du(x)}{dx}$$

Derivative exists iff both left right lim equal

$$f'(x) \equiv \frac{df}{dx} \equiv \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\} \equiv \lim_{h\to 0^+}\left[\frac{f^+ - f^-}{h}\right] \equiv \lim_{h\to 0^-}\left[\frac{f^+ - f^-}{h}\right]$$

$$f^+ \equiv f(x+h)$$

$$f^- \equiv f(x-h)$$

2.3. Differentiation[diff]for short

$$df(x) = f'(x) * dx$$

$$df(\xi) = f'(\xi) * d\xi$$

So diffis very convenience to deal with complicated vcase like func of func

$$df(g_x) = f'(\ ) * dg(\ )$$

2.4. Some derivatives

Expression of lim= lim of expression

For easy proofs we'll use **new notation** $f^+ \equiv f(x+h)$ **and** $L \equiv \lim_{h\to 0^\pm}$

Using new notation the derivative definitioncan be rewritten as

$$f'(x) \equiv \frac{L(f^+ f)}{L[h]}$$

2.4.1. Derivative of 1/x

$$f(x) = \frac{1}{x}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{-h}{x(x+h)}}{h} = \frac{-1}{x(x+h)} \xrightarrow{h\to 0} \frac{-1}{x^2}$$

$$\left[\frac{1}{x}\right]^{(\ )} = \frac{-1}{x^2}$$

2.4.2. Derivativef of Power

$$[x^2]^{(\ )} = [x * x]^{(\ )} = x' * x + x * x' = 2x,\ x' = 1$$

$$[x^3]^{(\ )} = [x * x^2]^{(\ )} = x' * x^2 + x * [x^2]^{(\ )} = x^2 + x * [2x] = 3x^2$$

Similarly

$$[x^n] = nx^{n-1}$$

$$[x^n] = nx^{n-1}$$

3. Taylor series

Consider Power series

$$P(x) = a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

$$P^{(0)}(x) = a_0 + a_1 + a_2 x^1 + 2a_2 x^1 + 3a_3 x^2 \cdots + n a_{n-1} x^{n-1}$$

$$P^{(1)}(x) = a_1 + 2a_2 x^1 + 2 * 3a_3 x + \cdots + n * (n-1) a_n x^{n-2}$$

$$P^{(2)}(x) = 2a_2 + \cdots + 2 * 3a_3$$

Let

$$x = 0$$

$$a_0 = P(0)$$

$$a_1 = P^{(1)}(0)$$

$$a_2 = \frac{1}{2!} P^{(2)}(0)$$

$$a_3 = \frac{1}{3!} P^{(3)}(0)$$

$$a_n = \frac{1}{n!} P^{(3)}(0)$$

So **Taylor series**

$$f(x) \approx f(0) + x f^{(1)} f(0) + \frac{1}{2!} x^2 f^{(2)}(0) + \frac{1}{3!} x^3 f^{(3)}(0) + \cdots$$

$$f(x) \approx f(0) + x f^{(1)} f(0) + \frac{1}{2!} x^2 f^{(2)}(0) + \frac{1}{3!} x^3 f^{(3)}(0) + \cdots$$

Let

$$x = z - a$$

then

$$x = 0 \Rightarrow z = a$$

Hence

$$(x) \Rightarrow (z - a)$$

$$f(x) \Rightarrow f(z - a)$$

$$f(0) \Rightarrow f(a)$$

$$f(z-a) \approx f(0) + (z-a) f^{(1)} f(a) + \frac{1}{2!}(z-a)^2 f^{(2)}(a) + \frac{1}{3!}(z-a)^3 f^{(3)}(a) + \cdots$$

$$f(z-a) \approx f(0) + (z-a) f^{(1)} f(a) + \frac{1}{2!}(z-a)^2 f^{(2)}(a) + \frac{1}{3!}(z-a)^3 f^{(3)}(a) + \cdots$$

4. Logarithmic, Natural Log &Exponential funtion

4.1. Logarithmic, Natural Log

Derivative of quotient is $\left[\frac{f(x)}{g(x)}\right]^{(\ )} = \frac{f'g - g'*f}{g^2}$

Derivative of function $\frac{1}{x}$ is $\frac{-1}{x^2}$

On the othersidewhat function whose deruvative is $\frac{1}{x}$ It is the **logarithmic function known as natural log**

$$[\log(x)]^{(\ )} \equiv \frac{1}{x}$$

4.2. Exponential funtion

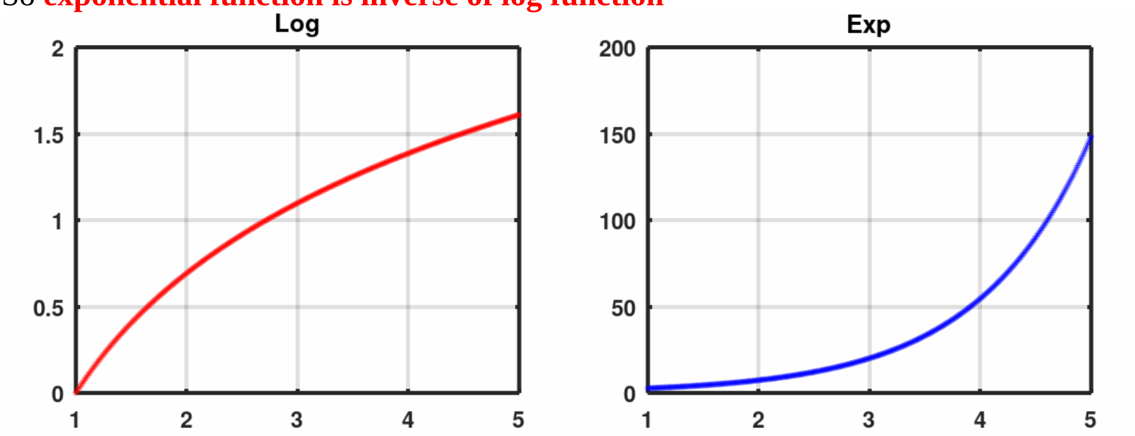
Every function has its **inverse version** define as

$$y = f(x) \Leftrightarrow x = f^{I}(y)$$

For example

$$y = f(x) = 2x \Leftrightarrow x = f^{I}(y) = \frac{y}{2}$$

So **exponential function is inverse of log function**



$$y = \log(x) \Leftrightarrow x = \exp(y) = e^{y}, e = 3.71828$$

4.3. Properties of function log and exp

$$y = \lg(x) \Leftrightarrow x = e^{y} = e^{\lg(x)}$$

and vice versa

$$\lg(e^{x}) = x$$

$$e^{(a+b)} = e^{a} * e^{b}$$

$$a + b = \lg(e^{a} * e^{b}) \Rightarrow \lg(e^{a}) + \lg(e^{b})$$

4.3.1. So log of product is sum of log

$$\lg(c * d) = \lg(c) + \lg(d)$$

4.3.2. log base a

$$y = a^x \Leftrightarrow x = la(y)$$

taking lg on both side

$$\lg y = x \lg a \Leftrightarrow x = \frac{\lg y}{\lg a}$$

$$la(y) = \frac{\lg(y)}{\lg(a)}$$

4.4. Derivative of product

$$f(x) = u(x) * v(x)$$

$$let \;\; f_x = f(x)$$

$$\lg(f_x) = \lg(f_x) + \lg(g_x)$$

$$\frac{df}{f} = \frac{du}{u} + \frac{dv}{v}$$

$$df = \frac{v * du + u * dv}{u * v}$$

Multiply x to both side

$$df * dx = \frac{v * dudx + u * dv * dx}{u * v}$$

$$f == \frac{u'*v + u * v'}{uv}$$

4.5. Derivative of quotient

$$f(x) = \frac{u(x)}{v(x)}$$

$$let \;\; f_x = f(x)$$

$$\lg(f_x) = \lg(f_x) - \lg(g_x)$$

$$\frac{df}{f} = \frac{du}{u} - \frac{dv}{v}$$

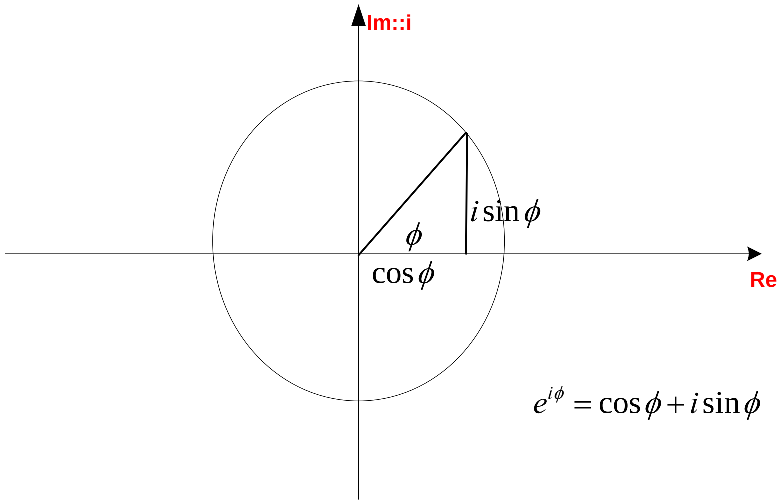
$$df = \frac{v * du - u * dv}{u * v}$$

Multiply x to both side

$$df * dx = \frac{v * dudx - u * dv * dx}{u * v}$$

$$f == \frac{u'*v - u * v'}{uv}$$

4.6. Euler's formula



$$e^{i\phi} = \cos\phi + i\sin\phi$$

5. Derivative of Trig function

Taking derivative of Euler formula to have

$$ie^{i\phi} = \cos'\phi + i\sin'\phi$$

$$i(\cos + i\sin\phi) = \cos'\phi + i\sin'\phi$$

$$(-\sin i\cos + i\sin\phi) = \cos'\phi + i\sin'\phi$$

equate Im and Re part to have

$$\cos'\varphi = \text{-}\sin$$

$$\sin'\phi = \cos$$

6. Sin Cos identities

$$\exp(a+b)=\exp(a)*\exp(b)$$
$$\cos(a+b)+i*\sin(a+b)=[\cos(a)+i*\sin(a)]*[\cos(b)+i*\sin(b)]$$
$$=[\cos(a)\cos(b)-\sin(a)\sin(b)]+i*[\sin(a)*\cos(b)+\sin(b)*\cos(a)]$$

Equate Re_part and Im_part to have

$$\exp(a+b)=\exp(a)*\exp(b)$$
$$\cos(a+b)+i*\sin(a+b)=[\cos(a)+i*\sin(a)]*[\cos(b)+i*\sin(b)]$$
$$=[\cos(a)\cos(b)-\sin(a)\sin(b)]+i*[\sin(a)*\cos(b)+\sin(b)*\cos(a)]$$

$$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$
$$\sin(a+b)=\sin(a)*\cos(b)+\sin(b)*\cos(a)$$

7. Derivative of Inv Trig function

The goal is to find Taylor series to compute Pi $\arctan(1)=\frac{\pi}{4}$

7.1. Derivative of arctan

$$y=\arctan x \Leftrightarrow x=\tan y$$

$$dx=d(\tan y)dy=d\left(\frac{\sin y}{\cos y}\right)=\frac{\cos^2 y+\sin^2 y}{\cos^2 y}dy=(1+\tan^2 y)dy=(1+x^2)dy \Rightarrow \frac{dy}{dx}=\frac{1}{1+x^2}$$

hence

$$f^{(1)}(x)=\frac{1}{1+x^2}=(1+x^2)^{-1} \Rightarrow f^{(1)}(0)=1$$
$$y^{(2)}=-2x(1+x^2)^{-2} \Rightarrow f^{(2)}(0)=0$$
$$y^{(3)}=-2(1+x^2)^{-2}+x(*) \Rightarrow f^{(2)}(0)=-2$$

Keep taking derivative for higher orderfor Taylor series to comput Pi values

All even order derivatives are ZERO thus only odd order

$$arf(x)=f(0)+xf^{(1)}(0)+\frac{x^2}{2!}f^{(2)}(0)+\frac{x^3}{3!}f^{(3)}(0)=x-\frac{x^3}{3}+\frac{x^5}{5}=\sum\left((-1)^k\frac{(x)^{2k+1}}{(2k+1)}\right)$$

7.2. Derivative of arcsin

ncase $x\in\left[0,\frac{\pi}{2}\right] \Rightarrow x>0, \sin x>0, \cos x>0, \tan x>0, \cot x>0$

$$y=\arcsin x \Leftrightarrow x=\sin y$$

$Asum \min g$ $\cos y>0$

$$dx=d(\sin y)=\cos y dy=\sqrt{1+\sin^2 y}dy \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1+\sin^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

$$\arcsin x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1*3}{2*4}\frac{x^5}{5}=$$

. **Using 2 Tthousands terms terms we have Pi= 3.141642651089887**

. Using 2, billion terms 000,000,000 we have :
3.141592658505056867568328016204759478569031000000000
Arcsin cannot be used due to the product become to large and cause overflow

$$\sin^{I} = a\arcsin = a\sin$$

$$\cos^{I} = a\cos$$

$$\tan^{I} = a\tan$$

$$y = a\sin x \Leftrightarrow x = \sin y \Rightarrow x^2 = \sin^2 y = 1 - \cos^2 y \Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$dx = d(\sin y) = \cos y * dy =$$

$$a\sin' x = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Similarly

$$y = a\cos x \Leftrightarrow x = \cos y \Rightarrow dx = -\sin y dy$$

$$x^2 = \cos^2 y = 1 - \sin^2 y \Rightarrow \sin^2 y = 1 - \cos^2 y \Leftarrow \sin y = \sqrt{1 - \cos^2 y}$$

$$dx = -\sqrt{1 - \cos^2 y} * dy = \sqrt{1 - x^2} * dy$$

$$a\cos'(x) = \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

$$y = a\tan x \Leftrightarrow x = \tan y \Rightarrow dx = (1 + \tan^2 y)dy = (1 + x^2)dy$$

$$a\tan' x = \frac{dy}{dx} = \frac{1}{1 + x^2}$$

8. Continous function and delta Dirac function

8.1. Continous function

A smooth function $f(x)$ is any curve for which $f(x)$ ~r0 (t) is continuous afor anyx t except possibly at the endpoints. Sine function is continuous function but step function $s(x)$ belowe is not
It's dis continuous at $x = a$ due to difference in leftright limit

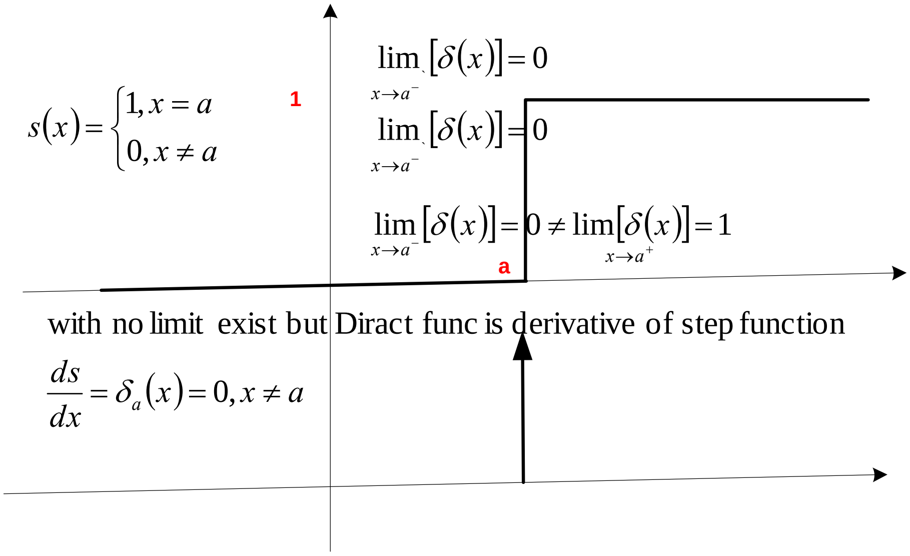
$$\lim_{x \to a^-} [f(x)] = 0$$

$$\lim_{x \to a^+} [f(x)] = 1$$

A smooth curve is any curve for which ~r0 (t) is continuous and ~r0 (t) 6= 0 for any t except possibly at the endpoints. A sine function is a smooth function

8.2. Delta Dirac function

Dhe delta Dirac function (or d distribution), also known as the unit impulsee Dirac delta function (or d distribution), also known as the unit impulsewidely used in digital technology i



$$s(x)=\begin{cases}1, x=a\\0, x\neq a\end{cases}$$

$$\lim_{x\to a^-}[\delta(x)]=0$$

$$\lim_{x\to a^-}[\delta(x)]=0$$

$$\lim_{x\to a^-}[\delta(x)]=0\neq\lim_{x\to a^+}[\delta(x)]=1$$

with no limit exist but Diract func is derivative of step function

$$\frac{ds}{dx}=\delta_a(x)=0, x\neq a$$

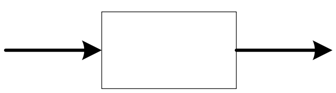
By defition Diract func is defines as derivative of unit step func

$$s(x)=\begin{cases}1,\ x>a\\0,\ otherwise\end{cases}$$ so by defitionof integral

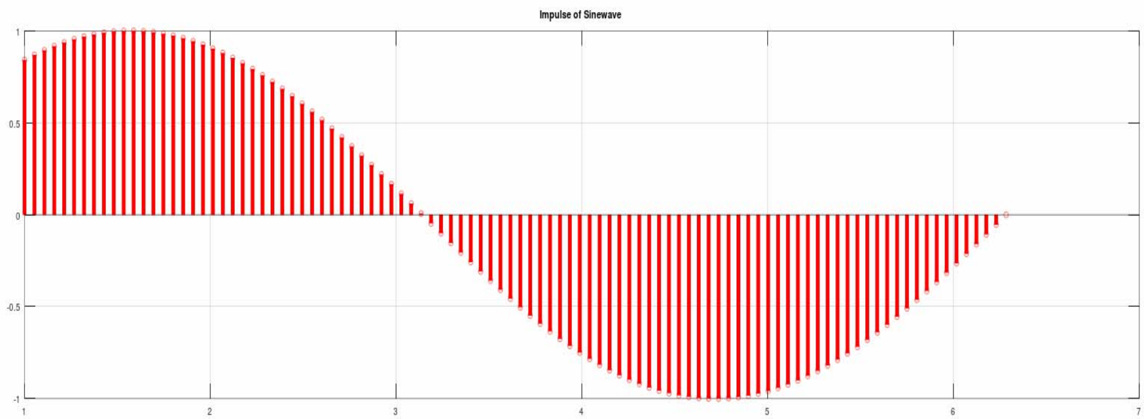
$$\int\delta(x)dx=1$$

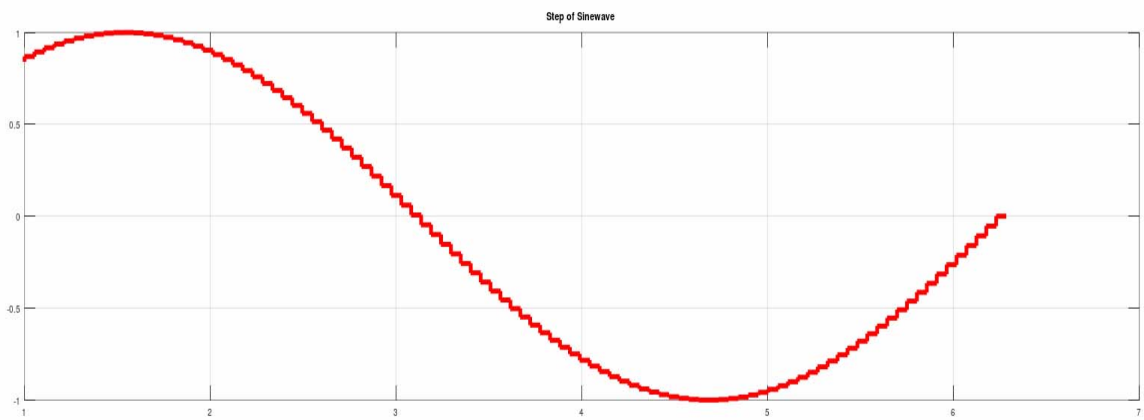
9. Analog & Digital signal

All natural signals in realword are continuousanalogd type like music soundsoundtime
But currently they are converted to digital type used in modern device like smart phone
A Device ADC[Analog Digital Converter]use to convertanalog to digital



The ADC his composed of 2 phases[1] sample to put out impulse response [2] Hold to convert impiulse response to stair response Time between 2 adjacent samples called sampling time
ADC quality is determineby 2 factor
[1]Sampling time:higher sampling time closer to analog signal
[2]resolution:how numberof bit in converteddigital signal , known as precision. higher precision closer to analog signa


Impulse of Sinewave
1
0.5
0
-0.5
-1
1
2
3
4
5
6
7


Step of Sinewave
1
0.5
0
-0.5
-1
1
2
3
4
5
6
7

Analog signals are continuous-time signals. *Discrete-time signals*, or *discrete signals* for short, take finite values of any real numbers, hence the difference of successive values is finite and is a real number.

Gradient&Hessian&Jacobian

Only Jacobian matrixmrow, ncolswork for m-vector function $(\mathbf{x}) \in \Re^m, \mathbf{x} \in^n$

$$\mathbf{J} = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

The others workfor scalar vlue function $f(\mathbf{x}) \in \Re, \mathbf{x} \in^n$

$$\mathbf{GRAD} = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$df = \sum \frac{\partial f}{\partial x_n} dx_n = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \bullet [dx_1 \quad dx_2 \quad \cdots \quad dx_n] = \nabla f(\mathbf{x}) \bullet d\mathbf{x}$$

$$\mathbf{HESSIAN} = \nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{bmatrix}$$

Taylor series

$$f(\mathbf{x}^* + a\mathbf{s}) = f(\mathbf{x}^* + a\mathbf{s}) + a\mathbf{s}^T \nabla f(\mathbf{x}^* + a\mathbf{s}) + \frac{1}{2} a^2 \mathbf{s}^T \nabla^2 f(\mathbf{x}^* + a\mathbf{s})\mathbf{s} + \cdots +$$

Theorem 1: Grad is direction of max change

$$f(\mathbf{x}^* + a\mathbf{s}) = f(\mathbf{x}^*) + a\mathbf{s}^T \nabla f(\mathbf{x}^*) + \frac{1}{2} a^2 \mathbf{s}^T \nabla^2 f(\mathbf{x}^*)\mathbf{s}$$

$$f(\mathbf{x}^* + a\mathbf{s}) = f(\mathbf{x}^*) + a\left[s|\nabla f^*|\cos(\langle \mathbf{s}, \nabla f^* \rangle)\right] + \frac{1}{2} a^2 \mathbf{s}^T \nabla^2 f(\mathbf{x}^*)\mathbf{s}, \ s = |\mathbf{s}|$$

The change $\left[s|\nabla f^*|\cos(\langle \mathbf{s}, \nabla f^* \rangle)\right]$is max when $cos(\langle \mathbf{s}, \nabla f^* \rangle) = 1$, i.e.$\mathbf{s}$ in direction of ∇f^*

QED

Rolle's Theorem

$f'(c) = 0, \ c \in (a,b)$

Case1 $f(x) = const \Rightarrow any\, x \in (a,b) f'(x) = 0$

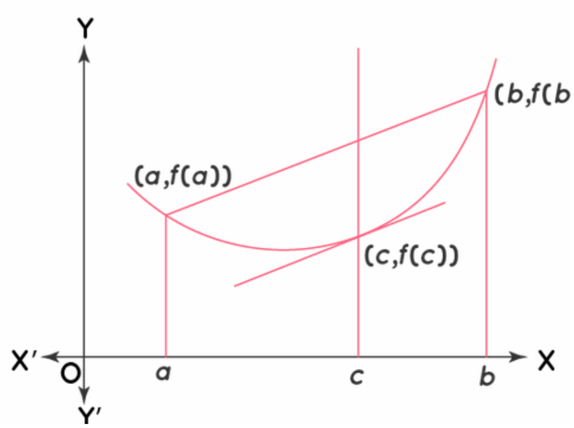
Case 2 $f(a) < f(b) \Rightarrow any\, x \in (a,b) f'(x) = 0$
$\exists d \in (a,b) : f(d) > f(a)$

Case1 $f(x) = const \Rightarrow any\ x \in (a,b) f'(x) = 0$

Mean value Theorem

Mean value Theorem

$$\phi'(c) = \frac{f(a) - f(b)}{a - b} c \in (a,b)$$



10. Integral

We have defition of derivative

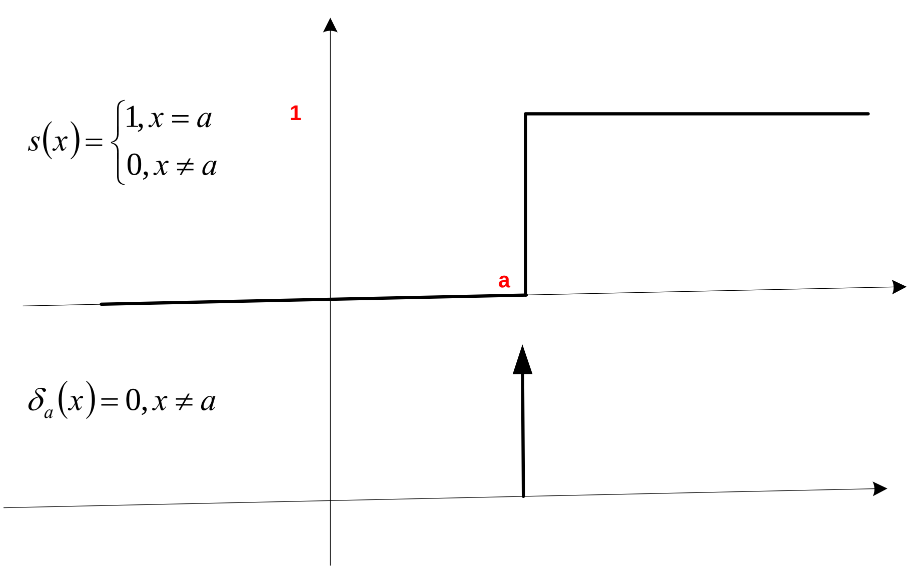
$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

We may have integral definition

$f = \sum df = \sum f'(x)dx$ also known as Riemann sum

$$f(x) = \int df(x) = \int f'(x)dx$$

10.1. Integral of Delta Dirac func



By defition Diract func is defines as derivative of unit step func

$s(x) = \begin{cases} 1, & x > a \\ 0, & otherwise \end{cases}$ so by defitionof integral

Dirac function has been used widely in real tekworld wehere its pulse width very small depending on application

$\int \delta(x)dx = 1$

Derivative of Delta Dirac is undefine but itsintegral is defined

$\int_{-\infty}^{+\infty} \delta(x)dx = 1$

We may see integral is an inverse of derivative $f = \int df$

So from derivative result above
We have

$\int x^n dx = \frac{x^{n+1}}{n+1}$

10.2. Integral by part

$d[uv] = vdu + udv$

$\int udv = uv - \int udv$

11. La-placeTransform

11.1. First derivative

Laplace transform of function $f(t), t \in (0,\infty)$ is defined as **function of complex var** $s = \sigma + j\omega, \omega = 2\pi f$

$\zeta[f(t)]f(s) = \int_0^\infty f(t)e^{-st}dt$ That's why Laplace transform analize an time domain function in frequency domainto make a lot easier in the world of science tek Like multipication of 2 function in frequncy domainknown as **convolution** coresponds to multiplication of these func in time domain

Impedance of capacitor and inductor can be derived using Laplace transform for circuit analysis

First to find Laplace transform for deriative as it's used for differential eq

$\zeta[f'(t)]f(s) = \int_0^\infty f'(t)e^{-st}dt$

bypart

$\int vdu = uv - \int udv$

$du = f'(t)dt \Rightarrow u = f(t)$

$v = e^{-st} \Rightarrow dv = -se^{-st}dt$

$\zeta[f'(t)] = [e^{-st} f(t)]_0^\infty + s\int_0^\infty f(t)e^{-st}dt$

$\zeta[f'(t)] = -f(0) = sF(s)$

$\zeta[f'(t)] = sF(s) - f(0)$

11.2. Second derivative

Let

$g(t) = f'(t) \Rightarrow g'(t) = f''(t)$

$G(s) = sF(s) - f(0)$

$\zeta[f''(t)] = \zeta[g'(t)] = sG(s) - g(0)$

$= s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0)$

$\zeta[f^{(2)(t)}] = s^2F(s) - sf(0) - f'(0)$

Similarly for **n-order derivative** $f^{(n)(t)}$

$$\zeta\left[f^{(n)}\right]=s^{n}F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-s^{n-3}f^{(2)}(0)-s^{0}f^{(n-1)}(0)$$

sum of power of s and derivative ordrer equal n-1

$$\zeta\left[f^{(n)}\right]=s^{n}F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-s^{n-3}f^{(2)}(0)-f^{(n-1)}(0)$$

$$\mathcal{L}\{f^{(n)}\}=s^{n}\mathcal{L}\{f\}-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$$

12. Capacitor Impedance

We have Capacitor law

$$i=C\frac{dv}{dt}\Rightarrow I(s)=C[sV(s)-v(0)]=sC*V(s)C-C*v(0)=sCV(s),\ v(0)=0$$

$$I(s)=sCV(s)$$

hence Imdedance of capacitor

$$Z_C=\frac{V(s)}{I(s)}=\frac{1}{sC}$$

13. Inductor Impedance

We have Inductor law

$$v=L\frac{di}{dt}\Rightarrow V(s)=L[sI(s)-i(0)]=sL*I(s)-Li(0)=sL*I(s)L*i(0),\ -i(0)=0$$

$$V(s)=sL*IV(s)$$

hence Imdedance of Inductorcapacitor

$$Z_L=\frac{V(s)}{I(s)}=sL$$

14. Brief Note on MLab

we can put our own M function in dir say "**c:\Octave\dkn**
and add this dir into the code using our Mfunction there with
addpah(:\Octave\dkn)

Octave, but not MLab, supports cmd line to run m code
Both index start at 1 like old laguage **fortran** used in **Linpack**[linear algebra packege not zero likein modern language
[] used for vector
() used for index start from 1
To create vector vof 10 values

```
For loop of i =1:10
v(i)=i
endfor
```

The best feature of both is support varying number of input arg to a fun yusind **varargin**

```
function y= chkk(varargin)
x_1= varargin{1} 1st ard
xx=varargin{nargin} last arg
end function
```

This the most impressed feature of Octave over MLab so I can do run and edit Mcode at the time so I'm not distracted by leave editor to IDE and fully focus pon the code
Bavo awsome Octave free and a lot better than MLab
Honestly Octave run at half speed of MLab but it doesn't matter to me during in development

Both support function of generral prams

We have to openMLab ide to run M code
Mlab use % for comment but Octave use #
Both use ; to supress printout var alue
Blk comment ust with {}
#{ open blk comment in Octave %{ in MLab
MLab use end for blk code of function for while if and last index of vector
Octave use endif endfor endfunction end used only as last index of vector
This is a better feature of Octave to chk blk code

This is better feature Octave as % is **renaider operator 3%2=1**
Both have the same syntax in block code start with statement and end at new statement orat end
MLab uses
if ... # start here
elseif #end of if
else #end of else if
end #end of else also end of the whole if
end #end
Only Octave, but not MLab, support unary oparator
n++; not print out value of n
x+=n print out value of x
Both require **name of file withfunction code same as function name**
To do qik chk of a funcon a a code say **abc.m** we have to put clear all at the top ofd code abc.mso we can put a funtion **xyz** inside otherwise we have error due to function name diff than filenamecodefile

15. Numerical Method

To present asimple and comprehensive approach for numerical

15.1. Definite Integral

It's easy to look at using definite integral to find area
Simply using

$$A = \sum (\ ) y_k * (x_{k+1} - x_k)$$

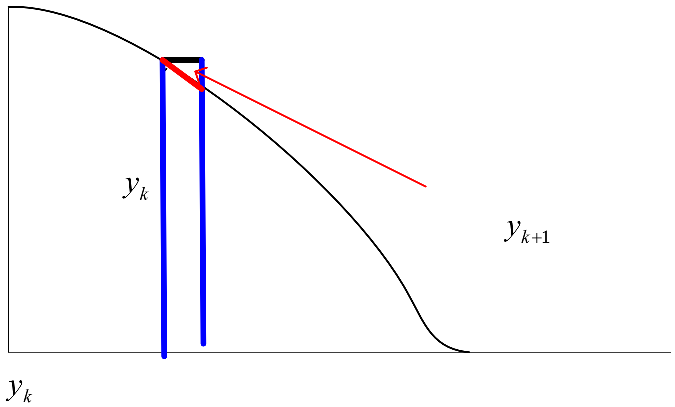
or more acurate

$$A = \sum \left[\frac{y_k + y_{k+1}}{2} * (x_{k+1} - x_k) \right]$$

To find area of quater of unit circle $x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}$

The exact are is $\frac{\pi}{4}$ **Error with Rectangle is1.659048425702050e-03**

while **Err with Trapzoid is 8.590660923892693e-05** so better by 2 order



15.2. Differential eq [DE]

Err with Rectang

No Err with Trap

For solution of Ddifferential Eq[**DE**], current numerical method've using Euler method relying on runge kutta 4th order method
Any differentialable function can be determine by tTe welknown Taylor series using its derivative from 1st to n-th order, the more the better
In current literature there're so many different methods like **Fourth-order Runge–Kutta method**
Adams-Bashforth 2-step method
Adams-Bashforth 3three-step method
Adams-Bashforth 4four-step method
Milne’s method
Adams-Moulton 2two-step method
Adams-Moulton 3three-step method
That's all I found You may find more
So it's obviouly Taylor series is the best way to solve DE and this note is about it

15.2.1. 1st Order DE

$$y' + ay = 0$$

$$\frac{y'}{y} = -a$$

$$y(t) = \exp(-at)$$

$$so$$

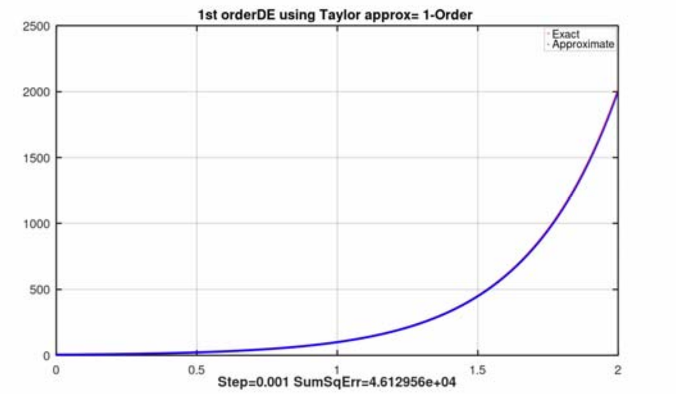
$$y^{(1)} = -ay$$

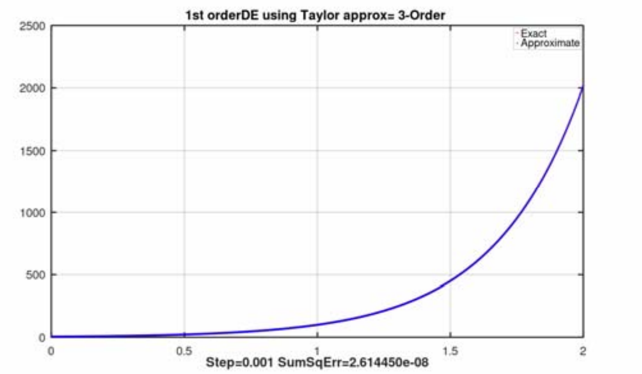
$$y^{(n)} = -ay^{(n-1)}, n \geq 2$$

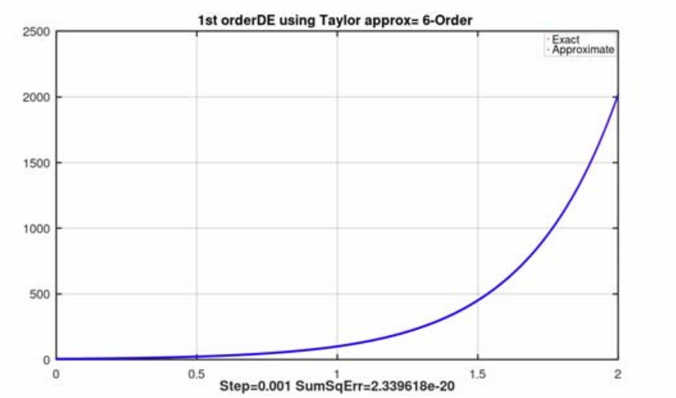
$$y^{(1)} = -y$$

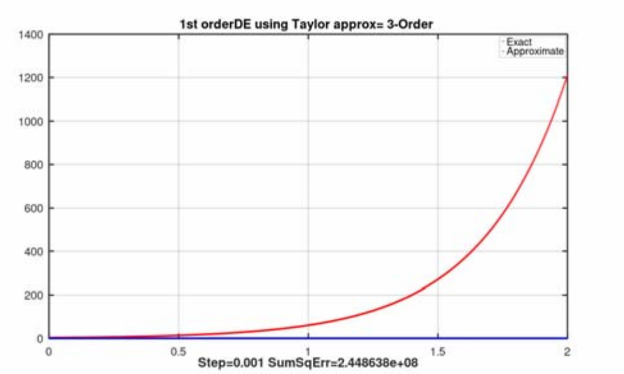
$$y^{(1)} = -y$$

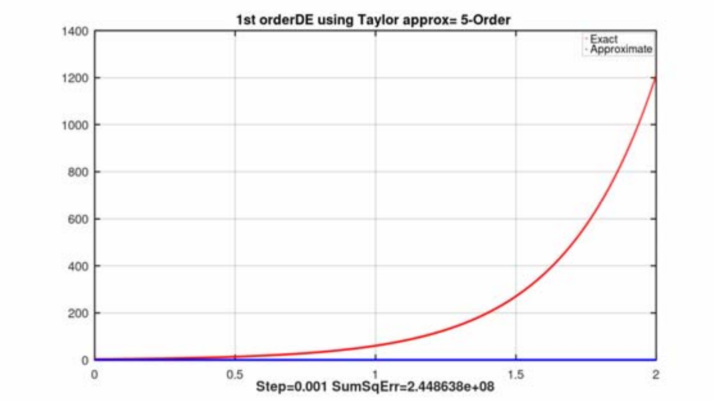
$$y^{(1)} = -y$$


1st orderDE using Taylor approx= 1-Order
Exact
Approximate
Step=0.001 SumSqErr=4.612956e+04


1st orderDE using Taylor approx= 3-Order
Exact
Approximate
Step=0.001 SumSqErr=2.614450e-08


1st orderDE using Taylor approx= 6-Order
Exact
Approximate
Step=0.001 SumSqErr=2.339618e-20


1st orderDE using Taylor approx= 3-Order
Exact
Approximate
Step=0.001 SumSqErr=2.448638e+08



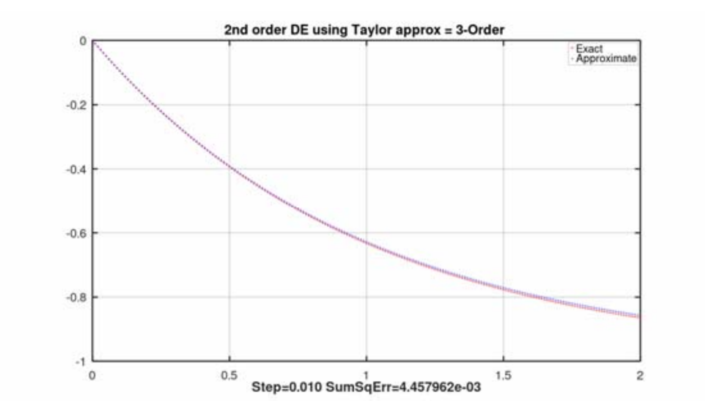
15.2.2. 2nd Order DE

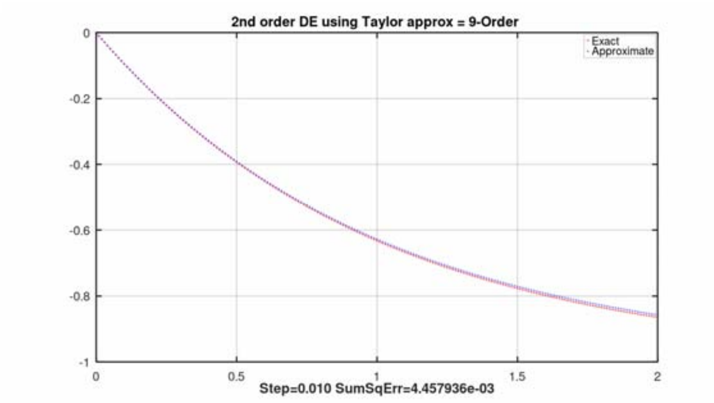
Remark 1

As y' is a significant component in Taylor series so its idnoranceand results in significant big err
So we must find it using integral as illustrated in example below

$$y''+ay'+b=0$$
$$y^{(2)}=-ay^{(1)}-b$$
$$Integrating\ for\ y^{(1)}$$
$$y^{(1)}=-ay-bt$$
$$Differentiating\ for\ y^{(n)},n\geq 3$$
$$y^{(3)}=-ay^{(2)}$$
$$y^{(n)}=-ay^{(n-2)},n\geq 3$$
$$s^2Y+asY+b=0$$
$$Y=\frac{-b}{s(s+a)}=\frac{A}{s}+\frac{B}{s+a}$$
$$A=-\frac{b}{a}$$
$$B=\frac{b}{a}$$
$$Y=\frac{b}{a}\left[\frac{1}{s}-\frac{1}{s+a}\right]$$
$$y(t)=-\frac{b}{a}\left[1-e^{-at}\right]$$

. **Err= 7.652528047444004e+01 If ignore** y'
. **Err = 4.457935854841033e-03 if y' used so Err is 3 orger larger**
So y' must be used for less err





$$y = \sin t$$

$$y^{(1)} = \cos t = \begin{cases} \sqrt{1-y^2}, & if \quad (t < \pi/2) \\ -\sqrt{1-y^2}, & else \end{cases}$$

$$y^{(2)} = -ay^{(1)} - b$$

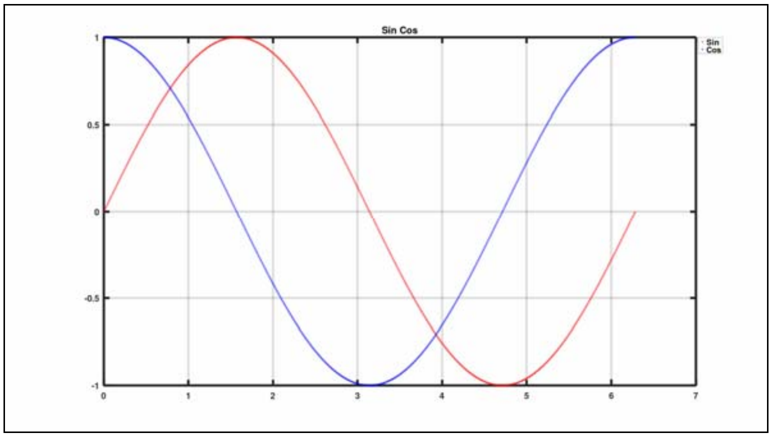
Integrating for $y^{(1)}$

$$y^{(1)} = -ay - bt$$

Differentiating for $y^{(n)}, n \geq 3$

$$y^{(3)} = -ay^{(2)}$$

$$y^{(n)} = -ay^{(n-1)}, n \geq 3$$



$$s^2Y + asY + b = 0$$

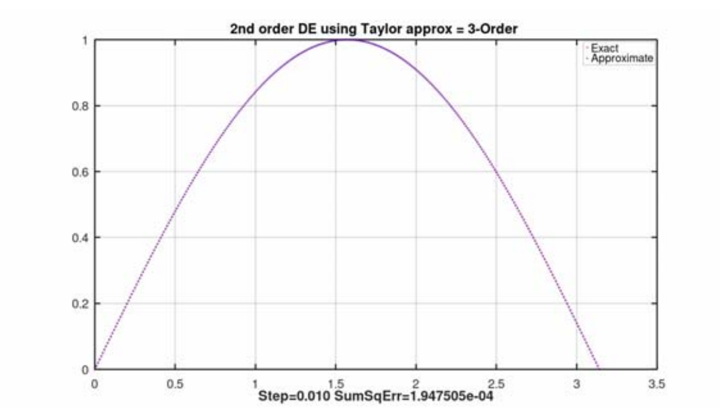
$$Y = \frac{-b}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$$

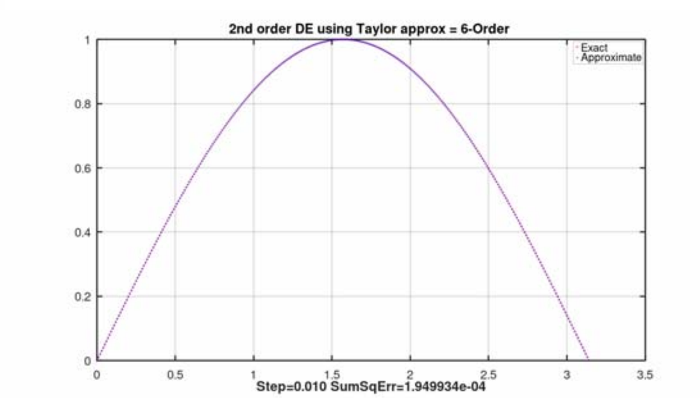
$$A = -\frac{b}{a}$$

$$B = \frac{b}{a}$$

$$Y = \frac{b}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right]$$

$$y(t) = -\frac{b}{a}\left[1 - e^{-at}\right]$$





If y' **inored Err= 157.08 else Err= 1.9475e-04 the Err is of 4 order larger**

Now we've got enoughto start solvinfg diff ed numerially
Westart with simple diff eq to illustrate the proedure
It's simple eoughto get it exact solution functio to check err between the exact and our numerical estimate
We have

$$y' = Ky$$

so

$$\frac{y'}{y} = K$$

multiplybothsides with dx to have

$$\frac{y'dx}{y} = Kdx$$

s

or

$$\frac{dy}{y} = Kdx$$

Taking integral to have

$$\ln y = Kx \Rightarrow y = f(x) = e^{Kx}$$

Using **Taylor series** to estimate y where $f^{(n)}(x)$n-order derivative

$$f(x) \approx f(0) + xf^{(1)}f(0) + \frac{1}{2!}x^2 f^{(2)}(0) + \frac{1}{3!}x^3 f^{(3)}(0) + \cdots \quad y = f(0) +$$

In our given case we have the sucessive derivative different by a factor K
In gerneral case a diff eq provide weher to get higher detrivative say fromdiff eq of 2nd order we can find 3rd derivative etc
We have

$$f^{(1)}(x) = Kf(x)$$

$$y(n) = Kf^{(n-1)}$$

so the algorithm for Taylor series estimate up to order N
Let y be estimate solution by Taylor seies , yd be the vector of derivative up to N values used in Taylor series
set initial value y(1)=f(0) index start at 1
yd(1)=K*y(1)
Just add term with detrivative to y up to N
so below is the Mcode addpath('C:\Octave\dkn'); # our factorial

```
N=1
#m=1;
K=2;
y0 = -6;        % Initial Condition
h = 0.001;% Time step
#A=0
tt = 0:h:2;       % tt goes from 0 to 2 seconds.
yy = zeros(size(tt)); % Preallocate array (good coding practice)
Err=zeros(size(tt));
z = y0*exp(K*tt);   % Exact solution (in general we won'tt know this)
yy(1) = y0;      % Initial condition gives solution at tt=0.
y(2)=y0;
END=length(tt)-1;% NOT rserved word "end" as lasr index of vector
yd=ones(1,N);
```

```
########################################################################################fa
for n=1:END
#####################################################################################
  yd(1)  =  K*(yy(n));  %  y=y0dot=K(A-y)-->yd(1)=-Ky  -->yd(2)=  -Ky_1dot  -->yd(3)= -Ky_2dot
   yy(n + 1) = yy(n) + h*yd(1);

for k=2:N
  % Keep taking derivative of y_dot
yd(k) = K*yd(k-1);
   yy(n + 1) +=   h^k/fa(k)*yd(k);
endfor #k
#####################################################################################

endfor #END
err = yy-z;
err2=err.^2;
mean_Sum_Sqr_Err = sum(err2)/END
Tool completed successfully
```

16. Result summary

Order of derivative	Mean SumSqr Err
1	0.1671
2	7.4490e-08
3	1.8619e-14
4	3.0144e-21
5	6.4282e-24
6	6.4282e-24
7	6.4282e-24
8	6.4282e-24
8	6.4282e-24
9	6.4282e-24

Note Just 2nd derivative err give sprisely little errhas derivative order 8 and 9 has same err

17. Clock Termination

um of power of s and derivative ordrer equal n - 1

begin with $s^n F(s)$ then substract all power s - - withderivative order from 0 , ie $f(0)$

then derivative order + + till (n - 1)

$$\zeta\left[f^{(n)}\right]=s^n F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-s^{n-3}f^{(2)}(0)-f^{(n-1)}(0)$$

A back plane widely used in a significant server in digital comunication system to make it adaptive to customer request with a primary board and optional secondary bard to customer need

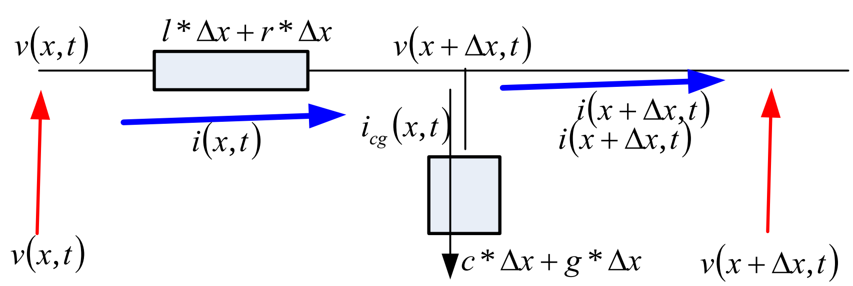
It's has nonle passive component like RLC butit's required clocok quality Some system ill behave due to bad clock quality

In my case to fix this problen for a main product XLI had it ioption card FTM(Frenquency Time measurement) it report double frequncy for poerline frequency of 50 Hz due to clock reflection per bad termination

I used transmission line to model clock trace and MLab simulation is matched with the real clock problem on

A model of transmission line is composed of inductor L in series[$l*\Delta x$] and capicitor in parallel[shunt] Cper lengthx $l*\Delta x$

Note r is distributed series resistance while g is distributed shunt conductanc so $g \neq \frac{1}{r}$



so we have v eq

$v(x) \equiv v(x,t)$

$i(x) \equiv i(x,t)$

$v=[l*\Delta x]*+[r*\Delta x]*i+v(x+\Delta x)$

$$v(x+\Delta x)-v(x)=-\Delta x*l*\left[\frac{\partial i(x,t)}{\partial t}+r*i(x,t)\right]$$

$$\frac{v(x+\Delta x,t)-v(x,t)}{\Delta x}=-l*\left[\frac{\partial i}{\partial t}+r*i\right]$$

let

$\Delta x \to 0$

to get

$$\frac{\partial v(x,t)}{\partial x}=-[l*+r*i]$$

$$\frac{\partial v(x,t)}{\partial x}=-\left[l*\frac{\partial i}{\partial t}+r*i\right]$$

L

and i eq

C, g

$$i_{cg}(x,t) = i(x,t) - i(x+\Delta x)$$

$$i_{cg}(x,t) = g*\Delta x*v(x+\Delta x,t) + c*\Delta x*\frac{\partial v(x+\Delta x,t)}{\partial t}$$

$$i(x,t) - i(x+\Delta x) = g*\Delta x*v(x+\Delta x,t) + c*\Delta x*\frac{\partial v(x+\Delta x,t)}{\partial t}$$

$$\frac{i(x,t) - i(x+\Delta x)}{\Delta x} = g + c*\frac{\partial v(x+\Delta x,t)}{\partial t}$$

$$i(x+\Delta x) = i(x) + g*v(x+\Delta x) \Rightarrow i(x+\Delta x) - = i(x) + g*v(x+\Delta x)$$

$$i(x,t) = [c*\Delta x]*\frac{\partial v}{\partial t} + [g*\Delta x]*v + i(x+\Delta x,t)$$

$$i(x+\Delta x,t) - i(x,t) = -\Delta x*\left[c*\frac{\partial v}{\partial t} + g*v(x,t)\right]$$

Similarly we have

$$\frac{\partial i(x,t)}{\partial x} = -\left[c*\frac{\partial v}{\partial t} + g*v\right]$$

Eventually we have an system of dif eq

$$\begin{cases} v(x,t) = -\left[l*\frac{\partial i}{\partial t} + r*i\right] \\ i(x,t) = -\left[c*\frac{\partial v}{\partial t} + g*v\right] \end{cases}$$

Simply simplify the eq system by Laplace transform

$$\begin{cases} v(x,t) = -\left[l*\frac{\partial i}{\partial t} + r*i\right] \Rightarrow V(s) = -l*[sI(s) + i(0)] - r*I(s) \\ i(x,t) = -\left[c*\frac{\partial v}{\partial t} + g*v\right] \\ i(x,t) = -\left[c*\frac{\partial v}{\partial t} + g*v\right] \end{cases}$$

$$v(x,t) = -l*\frac{\partial i}{\partial t} + r*\left[c*\frac{\partial v}{\partial t} + g*v\right]$$